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ON OPTIMAL ASSEMBLY OF SYSTEMS*

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ABSTRACT

We are concerned with the following reliability problem: A system has k different types of components. Associated with each component is a numerical value. Let $\{a^j\}$, $(j=1, \ldots, k)$, denote the set of numerical values of the k components. Let $R(a^1, \ldots, a^k)$ denote the probability that the system will perform satisfactorily (i.e., $R(a^1, \ldots, a^k)$ is the reliability of the system) and assume $R(a^1, \ldots, a^k)$ has the properties of a joint cumulative distribution function.

Now suppose $a_1^j \leq \ldots \leq a_n^j$ are n components of type j $(j=1, \ldots, k)$. Then n systems can be assembled from these components. Let N denote the number of systems that perform satisfactorily. N is a random variable whose distribution will depend on the way the n systems are assembled. Of all different ways in which the n systems can be assembled, the paper shows that EN is maximized if these n systems have reliability $R(a_i^1, \ldots, a_i^k)$ $(i=1, \ldots, n)$. The method used here is an extension of a well known result of Hardy, Littlewood, and Polya on sums of products. Furthermore, under certain conditions, the same assembly that maximizes EN minimizes the variance of N.

Finally, for a similar problem in reliability, it is shown that for a series system a construction can be found that not only maximizes the expected number of functioning modules, but also possesses the stronger property of maximizing the probability that the number of functioning modules is at least r, for each $0 \leq r \leq n$.

1. AN OPTIMAL ASSIGNMENT THEOREM

A well-known result appearing in Hardy, Littlewood, and Polya [1] assets if $a_1 \leq a_2 \leq \ldots \leq a_n$ and if $b_1 \leq b_2 \leq \ldots \leq b_n$, then

$$\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i b_{\psi(i)},$$

where ψ is any permutation of the integers 1, 2, . . ., n.

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Also in [1] is the generalization that for any $k \geq 2$ and if $0 \leq a_1^j \leq \ldots \leq a_n^j$, $j=1, \ldots, k$ then

$$\sum_{i=1}^{n} \prod_{j=1}^{k} a_i^j \geq \sum_{i=1}^{n} \prod_{j=1}^{k} a_{\psi_j(i)}^j,$$

where $\psi_1(i)=i$ $(i=1, \ldots, n)$ and $\psi_2, \ldots, \psi_k$ are any k permutations of $1, 2, \ldots, n$. Proof of both results follows from establishing the inequalities for the special case of $n=2$ and then resorting to a standard argument regarding pairwise interchanges. First let $n=2$ and $k=2$. Then

$$a_1b_1+a_2b_2-a_1b_2-a_2b_1=a_1(b_1-b_2)-a_2(b_1-b_2)=(a_1-a_2)(b_1-b_2) \geq 0,$$

since $a_1 \leq a_2$ and $b_1 \leq b_2$. Now assume the result is true for $n=2$ and $k=2, 3, \ldots, K-1$. Then consider

$$\sum_{i=1}^{2} \prod_{j=1}^{K} a_{\psi_j(i)}^j.$$

If for each $j=2, \ldots, K$, $\psi_j(1)=2$, $\psi_j(2)=1$, then

$$\prod_{j=2}^{K} a_{\psi_j(1)}^j = b_2 \text{ (say)}$$

and

$$\prod_{j=2}^{K} a_{\psi_j(2)}^j = b_1 \text{ (say)}$$

$$\leq b_2,$$

and, hence,

$$\prod_{j=1}^{K} a_1^j + \prod_{j=1}^{K} a_2^j = a_1^1 b_1 + a_2^1 b_2$$

$$\geq a_1^1 b_2 + a_2^1 b_1$$

$$= \prod_{j=1}^{K} a_{\psi_j(1)}^j + \prod_{j=1}^{K} a_{\psi_j(2)}^j.$$

If, for at least one j (say, $j=2$), $\psi_j(1)=1$, $\psi_j(2)=2$, then by the induction assumption

$$a_1^1\left[a_1^2 \prod_{j=3}^{K} a_{\psi_j(1)}^j\right]+\left[a_2^1 a_2^2 \prod_{j=3}^{K} a_{\psi_j(2)}^j\right]=a_1^1\left[\prod_{j=2}^{K} a_{\psi_j(1)}^j+\prod_{j=2}^{K} a_{\psi_j(2)}^j\right]+(a_2^1-a_1^1)\left[\prod_{j=2}^{K} a_{\psi_j(2)}^j\right] \leq$$

$$a_1^1\left[\prod_{j=2}^{K} a_1^j+\prod_{j=2}^{K} a_2^j\right]+(a_2^1-a_1^1)\left[\prod_{j=2}^{K} a_{\psi_j(2)}^j\right] \leq a_1^1\left[\prod_{j=2}^{K} a_1^j+\prod_{j=2}^{K} a_2^j\right]+(a_2^1-a_1^1)\prod_{j=2}^{K} a_2^j=\prod_{j=1}^{K} a_1^j+\prod_{j=1}^{K} a_2^j.$$

Hence the inequality holds for $n=2$, and all integer values of k.

Returning to arbitrary n, if $\psi_2, \ldots, \psi_k$ are not all equal to ψ_1 then there exist two values of i (say $i=1$, and $i=2$) such that $\psi_j(1) \leq \psi_j(2)$ does not hold for all $j=2, \ldots, k$. However, by considering the permutations $\psi_j'(j=2, \ldots, k)$ which are the same as ψ_j, $(j=2, \ldots, k)$ except that $\psi_j'(1) \leq \psi_j'(2)$ for every $j=2, \ldots, k$ it follows from the above result for $n=2$ that (keeping $\psi_1'(i)=i$)

$$\sum_{i=1}^{n} \prod_{j=1}^{k} a^j_{\psi_j(i)} \geq \sum_{i=1}^{n} \prod_{j=1}^{k} a^j_{\psi_j(i)}.$$

Hence, the original permutations could not be optimal.

Now let $R(x_1, \ldots, x_n)$ be any real-valued joint cumulative probability distribution function. For our purposes we want to prove the following extension.

THEOREM 1*: If $a_1^j \leq \ldots \leq a_n^j$ $(j=1, \ldots, k)$, then

$$\sum_{i=1}^{n} R(a_i^1, \ldots, a_i^k) \geq \sum_{i=1}^{n} R(a^1_{\psi_1(i)}, a^2_{\psi_2(i)}, \ldots, a^k_{\psi_k(i)}),$$

where $\psi_1(i)=i$ $(i=1, \ldots, n)$ and ψ_j $(j=2, \ldots, k)$ are any permutations of $1, 2, \ldots, n$.

PROOF:. As in the above proof we need only prove the theorem for $n=2$. Also, without loss of generality we can then take $a_1^j=0, a_2^j=1, j=1, \ldots, k$, and we can take R to be a discrete distribution function with mass only at the points $X_1, \ldots, X_{2^k}$ where X_m $(m=1, \ldots, 2^k)$ are all the points consisting of the k coordinates which are 0 or 1. Let C_m denote the probability mass at the point X_m. For any $\psi_2, \ldots, \psi_k$ we can write

$$R(a_1^1, a^2_{\psi_2(1)}, \ldots, a^k_{\psi_k(1)}) + R(a_2^1, a^2_{\psi_2(2)}, \ldots, a^k_{\psi_k(2)}) \leq \sum_{m=1}^{2^k} C_m + R(0, \ldots, 0) =$$

$$R(1, \ldots, 1) + R(0, \ldots, 0).$$

This inequality holds because the only point $X=(x_1, \ldots, x_k)$ that satisfies both of the conditions

$$x_1 \leq a_1^1, x_2 \leq a^2_{\psi_2(1)}, \ldots, x_k \leq a^k_{\psi_k(1)} \text{ and}$$

$$x_1 \leq a_2^1, x_2 \leq a^2_{\psi_2(2)}, \ldots, x_k \leq a^k_{\psi_k(2)} \text{ is the point}$$

$$X=(0, \ldots, 0).$$

Since $\psi_2, \ldots, \psi_k$ are arbitrary, the theorem is proved.

---

*This extension turns out to be a rediscovery of a special case of a result obtained by Lorentz [2]. We are indebted to the referee for simplifying our original proof.

2. APPLICATION TO RELIABILITY THEORY

We are concerned with a type of system that has k components. Associated with each component is a numerical value. Let $\{a^j\}$ $(j = 1, \ldots, k)$ denote the set of numerical values of the k components. We assume that $R(a^1, \ldots, a^k)$ is the probability that the system will perform satisfactorily (i.e., $R(a^1, \ldots, a^k)$ is the reliability of the system) where $R(a^1, \ldots, a^k)$ as a function of $a^1, \ldots, a^k$ has the properties of a joint cumulative distribution function. For example, if the system's performance depends on values of k random variables, $Y_1, \ldots, Y_k$, to the extent that the system performs satisfactorily, if and only if, $Y_j \leq a^j$, $j = 1,2, \ldots, k$, then if $Y_1, \ldots, Y_k$ have joint distribution $F(y_1, \ldots, y_k)$, then $R(a^1, \ldots, a^k) = F(a^1, \ldots, a^k)$ will have the assumed interpretation and properties.

Now suppose $a_1^j \leq \ldots \leq a_n^j$ are n components of type j $(j = 1, \ldots, k)$. Then n systems can be assembled from these components. Let N denote the number of systems that perform satisfactorily. N is a random variable whose distribution will depend on the way the n systems are assembled. A direct application of theorem 1 asserts:

THEOREM 2: Of all the $(n!)^{k-1}$ different ways in which the n systems can be assembled, EN is maximized if these n systems have reliabilities $R(a_i^1, \ldots, a_i^k)$ $(i = 1, \ldots, n)$.

PROOF: For any assembly defined by $\psi_1(i) = 1$, $\psi_j(i)$ $(j = 2, \ldots, n)$

$$EN = \sum_{i=1}^{n} R\left(a^1_{\psi_1(i)}, a^2_{\psi_2(i)}, \ldots, a^k_{\psi_k(i)}\right).$$

The previous result holds.

We can assert, also, the following result.

THEOREM 3: If $R(a^1_{\psi_1(i)}, \ldots, a^k_{\psi_k(i)}) \geq 1/2$ for every i and $\psi_2, \ldots, \psi_k$ then the same assembly that maximizes EN minimizes the variance of N.

PROOF: Since variance $N = \sum_{i=1}^{n} R(a^1_{\psi_1(i)}, \ldots, a^k_{\psi_k(i)})(1 - R(a^1_{\psi_1(i)}, \ldots, a^k_{\psi_k(i)}))$ the truth of the theorem need only be established for $n = 2$. That it is true for $n = 2$ follows from the following lemma:

LEMMA: If $1/2 \leq p_1 \leq q_1 \leq p_2 \leq 1$, $p_1 \leq q_2 \leq p_2$, and $p_1 + p_2 \geq q_1 + q_2$, then $p_1(1-p_1) + p_2(1-p_2) \leq q_1(1-q_1) + q_2(1-q_2)$.

PROOF: On subtracting the right hand member from the left, we get

$$\begin{aligned}
p_1(1-p_1) + p_2(1-p_2) - q_1(1-q_1) - q_2(1-q_2) &= p_1 + p_2 - p_1^2 - p_2^2 - q_1 - q_2 + q_1^2 + q_2^2 \\
&= q_1^2 - p_1^2 + q_2^2 - p_2^2 + (p_1 - q_1) + (p_2 - q_2) \\
&= (q_1 - p_1)(q_1 + p_1) + (q_2 - p_2)(q_2 + p_2) - (q_1 - p_1) - (q_2 - p_2) \\
&= (q_1 - p_1)[q_1 + p_1 - 1] + (q_2 - p_2)(q_2 + p_2 - 1) \\
&\leq (p_2 - q_2)(q_1 + p_1 - 1) + (q_2 - p_2)(q_2 + p_2 - 1) \\
&= (p_2 - q_2)(q_1 + p_1 - 1) - (p_2 - q_2)(q_2 + p_2 - 1)
\end{aligned}$$

$$= (p_2 - q_2)(q_1 + p_1 - q_2 - p_2)$$

$$= -(p_2 - q_2)(q_2 - q_1 + p_2 - p_1) \leq 0.$$

Some remarks are in order:

1. Theorems 2 and 3 will be applicable to special coherent structures, namely, those where the reliability function is not only monotone, but also a distribution function.

2. It should be stressed that the exact form of the distribution function need not be known, only that the reliability function can be assumed to be in the form of some distribution function.

3. ANOTHER APPLICATION OF THE HARDY, LITTLEWOOD, AND POLYA RESULT TO RELIABILITY

Consider the following model: A stockpile consists of n units of each of k different types of items. Associated with each item is a probability that the item will perform effectively. We denote by P_i^j, $j = 1, \ldots, k$, $i = 1, \ldots, n$, the probability that the ith unit of the type j will perform effectively. (This item will be referred to as the P_i^j item.) These probabilities are assumed to be independent and it is supposed that

$$P_1^j \leq P_2^j \leq \ldots \leq P_n^j, \qquad j = 1, \ldots, k.$$

From these nk items we must construct n modules, where a module consists of one of each of the k types of items. We say that a module functions if all of its k elements perform effectively. The problem is to construct the modules in such a way so as to maximize the probability of attaining at least $0 \leq r \leq n$ functioning modules.

Let us define the ith module to be that module containing the item P_i^1. If our objective was to maximize the expected number of functioning modules, then the Hardy, Littlewood and Polya result yields the solution. Namely, that the ith module consists of the items $P_i^1, P_i^2, \ldots, P_i^k$. We now show that this construction not only maximizes the expected number of functioning modules but it also possesses the stronger property of maximizing the probability that the number of functioning modules is at least r, for each $0 \leq r \leq n$.

THEOREM 4: The probability of obtaining at least r functioning modules is maximized, for each $r = 1, \ldots, n$, by letting the ith module consist of the items

$$P_i^1, P_i^2, \ldots, P_i^k, \qquad i = 1, \ldots, n.$$

PROOF: Consider any arbitrary construction of modules—call it C_1—whose 1st module consists of the items

$$P_1^1, P_{i_2}^2, P_{i_3}^3, \ldots, P_{i_k}^k,$$

where $i_2 \neq 1$, and suppose that the sth module of C_1 consists of the items

$$P_s^1, P_1^2, P_{j_3}^3, \ldots, P_{j_k}^k.$$

Now consider a different construction—call it C_2—whose 1st module consists of the items

$$P_1^1, P_1^2, P_{l_3}^3, \ldots, P_{l_k}^k,$$

and whose sth module consists of the items

$$P_s^1, P_{i_2}^2, P_{m_3}^3, \ldots, P_{m_k}^k,$$

where

$$l_r = \min(i_r, j_r), \qquad m_r = \max(i_r, j_r), \qquad r = 3, \ldots, k.$$

Suppose further that the remaining modules at C_2 are identical to those at C_1. The probability that either the 1st or the sth module of C_1 functions is

$$P_1^1 \cdot P_{i_2}^2 \cdot P_{i_3}^3 \cdot \ldots \cdot P_{i_k}^k + (1 - P_1^1 \cdot P_{i_2}^2 \cdot P_{i_3}^3 \cdot \ldots \cdot P_{i_k}^k)(P_s^1 \cdot P_1^2 \cdot P_{j_3}^3 \cdot \ldots \cdot P_{j_k}^k),$$

while the corresponding probability under C_2 is

$$P_1^1 \cdot P_1^2 \cdot P_{l_3}^3 \cdot \ldots \cdot P_{l_k}^k + (1 - P_1^1 \cdot P_1^2 \cdot P_{l_3}^3 \cdot \ldots \cdot P_{l_k}^k)(P_s^1 \cdot P_{i_2}^2 \cdot P_{m_3}^3 \cdot \ldots \cdot P_{m_k}^k).$$

As the number of the other $n-2$ modules which function is stochastically equal under C_1 or C_2 it follows by the Hardy, Littlewood, and Polya result that the number of functioning modules is stochastically larger under C_2 than it is under C_1. Hence we need only consider constructions whose first module contains both P_1^1 and P_1^2. Similarly we can show that we need only consider constructions whose first module is $P_1^1, P_1^2, \ldots, P_1^k$. Repeating this argument on the other modules completes the proof.

We make the following remarks:

1. The main reason we have considered this problem is that often the "total system" works if, and only if, at least r of the modules function.

2. If we suppose that $k = 2$ and that a module functions if at least one of the items in the module performs effectively then a proof similar to the above shows that the construction stochastically maximizing the number of functioning components is the one whose ith module consists of the items P_i^1, P_{n+1-i}^2.

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DETERMINING OPTIMAL GROWTH PATHS IN LOGISTICS OPERATIONS*

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ABSTRACT

This paper considers a logistics system modelled as a transportation problem with a linear cost structure and lower bounds on supply from each origin and to each destination. We provide an algorithm for obtaining the growth path of such a system, i.e., determining the optimum shipment patterns and supply levels from origins and to destinations, when the total volume handled in the system is increased. Extensions of the procedure for the case when the costs of supplying are convex and piecewise linear and for solving transportation problems that are not in "standard form" are discussed. A procedure is provided for determining optimal plant capacities when the market requirements have prespecified growth rates. A goal programming growth model where the minimum requirements are treated as goals rather than as absolute requirements is also formulated.

1. INTRODUCTION

This paper considers a logistics system modelled as a "transportation problem" [1, 4, 7] to minimize the total costs of transporting a homogenous good from a set of origins to a set of destinations. We are interested in the *growth path* of such a system, i.e., how the total cost, supplies and demands at origins and destinations, and the pattern of optimal shipments change when the total volume handled in the system increases subject to lower bounds on each origin's supply and each destination's demand. Such information would prove valuable in planning the system's short and long run resource requirements (facilities, manpower, financial needs, etc.).

This analysis is also useful in production-marketing-distribution systems; for instance consider a firm that wishes to expand and has to decide which factory capacities have to be increased and which additional markets should be sought after. We assume that it decides to maintain *at least* its current levels of production in the factories and supplies to markets.† The 'cost' of supplying from a factory to a warehouse would then involve production, transportation and promotion costs, and the revenue from selling the product. We assume that the net costs (or revenues) are *proportional* to the amounts involved so that we can obtain a net per unit cost (or negative per unit profit) for every factory-market

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†In controlled economies, the national government normally decides on such quotas.

combination. The growth path would then describe the behavior of the optimum solution (minimizing total cost or equivalently maximizing total profit) as the total sales volume is increased. Job shops that are already committed to certain jobs and shop personnel would also benefit from such analysis in deciding which additional jobs to take up and in overtime/hiring decisions.

In [3], Charnes, Glover and Klingman consider the "Lower Bounded and Partial Upper Bounded Distribution Model" which is the same as the problem defined earlier, except that they assume the total volume handled in the system to be a given number. Consequently, in order to trace the growth path using their methodology, one would have to solve a sequence of problems with different values for this parameter. The procedure considered in this paper, however, is a more direct procedure for determining the growth path using the operator theory of parametric programming for the transportation problem [8, 9].

In section 2 we formulate and solve the growth path problem and provide numerical examples. Section 3 considers extensions of this basic approach.

2. AN OPERATOR THEORETIC ALGORITHM FOR DETERMINING GROWTH PATHS

We consider below the "lower bounded and constrained total volume transportation problem" T by defining the index sets:

(1) $I = \{1,2, \ldots m\}$, the set of origins, and

(2) $J = \{1,2, \ldots n\}$, the set of destinations.

For $i \epsilon I$ and $j \epsilon J$, we define

x_{ij} = amount shipped from origin i to destination j,
c_{ij} = unit cost of shipping from origin i to destination j,
$a_i(b_j)$ = minimum amount to be supplied from origin i (to destination j) (assumed to be non-negative),
$S_i(T_j)$ = surplus amount supplied from origin i (to destination j) and
$g_i(h_j)$ = unit cost of supplying from origin i (to destination j).

Then we shall define the problem P as

(3) $$\text{Minimize } Z = \sum_{i \epsilon I} \sum_{j \epsilon J} c_{ij} x_{ij} + \sum_{i \epsilon I} g_i(a_i + S_i) + \sum_{j \epsilon J} h_j(b_j + T_j),$$

subject to the constraints:

(4) $$\sum_{j \epsilon J} x_{ij} = a_i + S_i \qquad \text{for } i \epsilon I,$$

(5) $$\sum_{i \epsilon I} x_{ij} = b_j + T_j \qquad \text{for } j \epsilon J,$$

(6) $$\sum_{i \epsilon I} \sum_{j \epsilon J} x_{ij} = K,$$

(7) $x_{ij} \geq 0$ for $i \in I$ and $j \in J$,

(8) $S_i \geq 0$ for $i \in I$, and

(9) $T_j \geq 0$ for $j \in J$,

where x_{ij}, S_i, and T_j are the decision variables. Constraint (4) states that the total amount supplied from origin i to all destinations should equal the total amount available at i which is $a_i + S_i$. Since $S_i \geq 0$ by (8), the total amount supplied from i is at least equal to its lower bound a_i. Similar comments apply to destinations $j \in J$. Constraint (6) requires the total volume handled in the transportation system must be equal to K. The objective function minimizes the total cost consisting of the transportation costs and the costs for supplying from the origins and to the destinations.

From (4) and (5) the objective function (3) can be seen to be equivalent to

(10) $$\text{Min } Z = \sum_{i \in I} \sum_{j \in J} c'_{ij} x_{ij} \text{ where } c'_{ij} = c_{ij} + g_i + h_j.$$

We, however, prefer to retain the objective function (3), since this form is more suitable when the costs of supplying from the origins and to the destinations are not necessarily proportional, so that $g_i(\cdot)$ and $h_j(\cdot)$ represent the respective cost functions of the arguments.

The 'growth path' problem consists of studying the variations in Z, $\{x_{ij}\}$, $\{S_i\}$, and $\{T_j\}$ as a function of K. We now determine the conditions on K so that the system (4)–(9) is feasible.

Summing the Equations (4) for $i \in I$ and subtracting from (6), we can replace Equation (6) by

(11) $$K - \sum_{i \in I} a_i = \sum_{i \in I} S_i.$$

From (8) it then follows that

(12) $$K \geq \sum_{i \in I} a_i = K_a \quad \text{(say)}.$$

Similarly from (5), (6), and (9) it follows that

(13) $$K - \sum_{j \in J} b_j = \sum_{j \in J} T_j, \text{ and}$$

(14) $$K \geq \sum_{j \in J} b_j = K_b \quad \text{(say)}.$$

In this section, we will assume that

(A1) $$K_a = K_b = K_o \quad \text{(say)}.$$

Later in this section, we show how this assumption can be relaxed. From (12), (14) and (A1), to assure

feasibility for (4)–(9) we need to assume:

(A2) $$K \geq K_o = \sum_{i \epsilon I} a_i = \sum_{j \epsilon J} b_j.$$

To bring the problem (3)–(9) to 'standard form', we convert it to a capacitated transportation problem [4, 7] (bounded variable transportation problem) with $(m+1)$ origins and $(n+1)$ destinations. Adding N (a large positive number) to both sides of (4) and (5) and rearranging, we get

(15) $$\sum_{j \epsilon J} x_{ij} + (N - S_i) = a_i + N \qquad \text{for } i \epsilon I, \text{ and}$$

(16) $$\sum_{i \epsilon I} x_{ij} + (N - T_j) = b_j + N \qquad \text{for } j \epsilon J.$$

Similarly (3), (11), and (13) become

(17) $$Z = \text{Min} \sum_{i \epsilon I} \sum_{j \epsilon J} c_{ij} x_{ij} - \sum_{i \epsilon I} g_i(N - S_i) - \sum_{j \epsilon J} h_j(N - T_j) + \sum_{i \epsilon I} g_i(N + a_i) + \sum_{j \epsilon J} h_j(N + b_j),$$

(18) $$\sum_{i \epsilon I} (N - S_i) = mN + \sum_{i \epsilon I} a_i - K, \text{ and}$$

(19) $$\sum_{j \epsilon J} (N - T_j) = nN + \sum_{j \epsilon J} b_j - K.$$

Defining,

(20) $$x_{i,n+1} = N - S_i,\ a_i' = a_i + N \text{ and } c_{i,n+1} = -g_i \qquad \text{for } i \epsilon I,$$

(21) $$x_{m+1,j} = N - T_j,\ b_j' = b_j + N \text{ and } c_{m+1,j} = -h_j \qquad \text{for } j \epsilon J,$$

(22) $$a_{m+1} = nN + \sum_{i \epsilon I} b_j - K,\ b_{n+1} = mN + \sum_{i \epsilon I} a_i - K,$$

(23) $$c_{m+1,n+1} = M \qquad \text{(a large positive number)},$$

(24) $$Z_o = \sum_{i \epsilon I} g_i(N + a_i) + \sum_{j \epsilon J} h_j(N + b_j)$$

(25) $$I' = I \cup \{m+1\} \qquad \text{and } J' = J \cup \{n+1\};$$

(17)–(19), (15)–(16), (7–9) become

(26) $$\text{Min } Z = Z_o + \sum_{i \epsilon I'} \sum_{j \epsilon J'} c_{ij} x_{ij}$$

subject to the constraints:

$$\sum_{j \in J'} x_{ij} = a'_i \qquad \text{for } i \in I', \tag{27}$$

$$\sum_{i \in I'} x_{ij} = b'_j \qquad \text{for } j \in J', \tag{28}$$

$$x_{ij} \geq 0 \qquad \text{for } i \in I' \text{ and } j \in J', \tag{29}$$

$$x_{i,n+1} \leq N \qquad \text{for } i \in I, \text{ and} \tag{30}$$

$$x_{m+1,j} \leq N \qquad \text{for } j \in J. \tag{31}$$

Since M is assumed to be a large positive integer, setting $c_{m+1,n+1} = M$ would lead to $x_{m+1,n+1} = 0$ in any optimal solution; similarly the conditions $x_{m+1,j} \geq 0$, $x_{i,n+1} \geq 0$ imply $S_i \leq N$ and $T_j \leq N$ which are no "real" constraints since N is very large.*

Rather than formulating the problem P as (26)–(31), we could have formulated it as a "transportation scheme" [10, 7 p. 250] by defining an additional origin $(m+1)$ and destination $(n+1)$ as before and defining $x'_{i,n+1} = -S_i \leq 0$ with unit cost $c_{i,n+1} = -g_i$ for $i \in I$ and $x'_{m+1,j} = -T_j \leq 0$ with unit cost $c_{m+1,j} = -h_j$ for $j \in J$.† We have preferred to use the capacitated transportation formulation since more readers are likely to be familiar with it than the transportation scheme and since the operator theory in [8, 9] is developed for the former.

We now consider the effect on the formulation (26)–(31) of changing the total volume shipped from K to $K + \delta$ (δ is assumed to be nonnegative). (Subsequently we will be studying the optimal solution for all nonnegative values of K so that it is not necessary to investigate the effect of changing K to $K - \delta$ separately though this extension can be easily done.) From (20)–(25) the only effect of this change a_{m+1} to $a_{m+1} - \delta$ and b_{n+1} to $b_{n+1} - \delta$. Suppose we have the optimum for the problem (26)–(31) for the value of $K = K_o$. Then the optimum solution for any other $K > K_o$ can be obtained by applying the *cell rim operator* $\delta \mathbf{R}^-_{m+1,n+1}$ [8, 9] with $\delta = K - K_o$. (This operator transforms the optimum solution of problem P into that of a problem P^- whose 'data' (i.e., c_{ij}, a_i and b_j) are the same as that of P except that $a^-_{m+1} = a_{m+1} - \delta$ and $b^-_{n+1} = b_{n+1} - \delta$). The above references provide constructive algorithms for finding the optimum solutions of all values of $\delta \geq 0$.

We now provide a method of finding a *basic optimum solution* to (26)–(31) for $K = K_o$ by using a transportation algorithm (rather than a capacitated transportation algorithm). From (11) and (A2),

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*The formulation (26)–(31) is close to that of problem (2) in [3] for the case $g_i = 0$ for $i \in I$ and $h_j = 0$ for $j \in J$. However, as will shortly be seen, our formulation is better suited to solving the "growth path problem."

†The formulation (26)–(31) is equivalent to Szwarc's transportation scheme, in the sense that there is a one to one correspondence between the feasible solutions to the two formulations (i.e., $x_{ij} = x'_{ij}$ for $i \in I$, $x_{i,n+1} = N + x'_{i,n+1}$ for $i \in I$ and $x_{m+1,j} = N + x'_{m+1,j}$ for $j \in J$, etc. where the 's denote feasible solution to the transportation scheme). In fact, every transportation scheme (i.e., $x'_{ij} \leq 0$ for $(i, j) \in \Psi$ say) can be converted to an equivalent capacitated transportation problem by defining $x_{ij} = x'_{ij} + N$ for $(i, j) \in \Psi$ ($x'_{ij} \leq 0 \Leftrightarrow x_{ij} \leq N$ and $x_{ij} \geq 0 \Rightarrow x'_{ij} \geq -N$ which is no 'real' constraint since N is assumed to be very large) and suitably modifying c_{ij}, a_i, and b_j. (For this reason, the formulation in [3] for minimizing $\sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}$ subject to $\sum_{j \in J} x_{ij} \geq a_i$ for $i \in I$ and $\sum_{i \in I} x_{ij} = b_j$ for $j \in J$ and $x_{ij} \geq 0$ for $i \in I$ and $j \in J$ is equivalent to the one in [7, p. 259]).

$\sum_{i \in I} S_i = 0$ since $K = K_o$ for the problem under consideration. Similarly $\sum_{j \in J} T_j = 0$. From (8) and (9) it then follows that $S_i = 0$ for $i \in I$ and $T_j = 0$ for $j \in J$. From (20) and (21) we obtain $x_{i,n+1} = N$ for $i \in I$ and $x_{m+1,j} = N$ for $j \in J$. (As usual, $x_{m+1,n+1} = 0$). Substituting this solution $x_{i,n+1}$ for $i \in I'$ and $x_{m+1,j}$ for $j \in J'$, the formulation (26)–(31) (for this particular problem with $K = K_o$) becomes:

$$\text{Min } Z = \left[Z_o - \sum_{i \in I} g_i N - \sum_{j \in J} h_j N \right] + \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} \tag{32}$$

subject to the constraints:

$$\sum_{j \in J} x_{ij} = a_i \qquad \text{for } i \in I, \tag{33}$$

$$\sum_{i \in I} x_{ij} = b_j \qquad \text{for } j \in J, \text{ and} \tag{34}$$

$$x_{ij} \geq 0 \qquad \text{for } i \in I \text{ and } j \in J. \tag{35}$$

The problem (32)–(35) is a transportation problem with mn variables and can be solved by known methods [1, 4, 7]. This optimum solution together with the known values of $x_{i,n+1}$ and $x_{m+1,j}$ provides an optimum solution to (26)–(31). However to construct a *basic* optimum solution,* we use the following result.

THEOREM 1: Let B denote the optimum basis for the problem (32)–(35) with u_i and v_j as the associated dual variables [4, 7]. Define the indices p, q, such that

$$v_{n+1} = -(u_p + g_p) = \max_{i \in I} [-(u_i + g_i)], \text{ and} \tag{36}$$

$$u_{m+1} = -(v_q + h_q) = \max_{j \in J} [-(v_j + h_j)]. \tag{37}$$

Then $B' = B + \{(p, n+1)\} + \{(m+1, q)\}$ is an optimal basis to the problem (26)–(31) with the dual solutions u_i for $i \in I'$ and v_j for $j \in J'$.

PROOF: We will first show that B' is a basis. Since B is a basis for the tableau with m rows and n columns, it has $m+n-1$ cells. Since $B' = B + \{(p, n+1)\} + \{(m+1, q)\}$ the number of basis cells in B' is $m+n+1$. There is at least one cell of B in rows $1, 2, \ldots m$ and columns $1, 2, \ldots n$. Since $(p, n+1)$ and $(m+1, q)$ are, in addition, present in B', the $(m+1)$st row and $(n+1)$st column also have a cell each. Finally there can be no cycles in B', since B does not have any cycles and the cells $(p, n+1)$, $(m+1, q)$ are unique in their column (row). Thus B' is a basis.

To show that B' is optimal, we have to establish that

$$c_{ij} = u_i + v_j \qquad \text{for } (i, j) \in B', \tag{38}$$

*Representing the problem (26)–(31) as a tableau with $m+1$ rows and $n+1$ columns, a *basis* B' is a set of $m+n+1$ cells (i.e., index pairs (i, j) with row $i \in I'$ and column $j \in J'$) such that there is at least one cell in every row and every column with no "cycles" (stepping stone tours). A solution $\{x_{ij}\}$ is *basic* if $(i, j) \notin B'$ implies $x_{ij} = 0$ or its upper bound.

(39) $c_{ij} \geq u_i + v_j$ for $(i,j) \notin B'$ such that $x_{ij} = 0$, and

(40) $c_{ij} \leq u_i + v_j$ for $(i,j) \notin B'$ such that x_{ij} is at its upper bound.

Since B is optimal to (32)–(35), (38) holds for the cells in B. For the cell $(p, n+1)$, $c_{p,n+1} = -g_p$ from (20). From (36), $-g_p = u_p + v_{n+1}$ so that $c_{p,n+1} = u_p + v_{n+1}$ thus satisfying (38). Similarly (38) holds for the basic cell $(m+1, q)$.

For the cells in $I \times J$, (39) holds since B is optimal. The solution to (26)–(31) was constructed so that $x_{i,n+1} = N$ for $i \in I$ and $x_{m+1,j} = N$ for $j \in J$. Thus these cells are at their upper bounds. For the cells $(i, n+1)$ with $i \in I$, from (36) it follows that

$$-(u_i + g_i) \leq v_{n+1}, \text{ or}$$

$$c_{i,n+1} = -g_i \leq u_i + v_{n+1}$$

satisfying (40) for these cells. Similarly (40) holds for $(m+1, j)$ for $j \in J$. For the cell $(m+1, n+1)$, $x_{m+1,n+1} = 0$ so that condition (39) trivially holds, since $c_{m+1,n+1} = M$ from (23).

We provide below an algorithm for finding the growth path. The first phase of this algorithm determines an optimum basic solution for $K = K_o$. The second phase finds the optimum solutions for $K_o \leq K \leq K_o + N$ by applying the operator $\delta \mathbf{R}^{-}_{m+1,n+1}$. (Recall that N and consequently the upper limit for K can be made arbitrarily large.) We denote by Ψ the set of cells (i, j) which have their respective x_{ij} bounded from above. From (30) and (31),

(41) $$\Psi = \{(i, n+1) \quad \text{for } i \in I\} \cup \{(m+1, j) \quad \text{for } j \in J\}.$$

Since only the cells in Ψ are bounded from above, we can somewhat simplify the statements of the algorithms in [8, 9] for applying $\delta \mathbf{R}^{-}_{m+1,n+1}$. This simplified version is given below and illustrated with an example. The reader should refer to [8, 9] for the proofs involved in Phase 2.

ALGORITHM 1: For finding the growth path—i.e., to find optimal solution of (26)–(31) for all $K_o \leq K \leq K_o + N$.

Phase 1: For finding a basic optimal solution for (26)–(31) for $K = K_o$.

1. Solve (32)–(35) using the MODI method [4, 7] (primal transportation algorithm). Let Z denote the optimum value of the objective function with basis B. Define the optimum primal solution $X = \{x_{ij}\}$ to (26)–(31) the same as that of (32)–(35) for $i \in I$ and $j \in J$. Set $x_{i,n+1} = N$ for $i \in I$, $x_{m+1,j} = N$ for $j \in J$ and $x_{m+1,n+1} = 0$. Determine the indices (p, q) and the duals v_{n+1} and u_{m+1} using (36)–(37). Set $B' = B + \{(p, n+1)\} + \{(m+1, q)\}$ as the corresponding optimum basis.

Phase 2: For applying the operator $\delta \mathbf{R}^{-}_{m+1,n+1}$ for any $0 \leq \delta \leq N$.

2. Set $k = 1$ (iteration number). $X_1 = X$, $B_1 = B'$, $Z_1 = Z$, $u_{i,1} = u_i$ for $i \in I'$, $v_{j,1} = v_j$ for $j \in J'$.

3. Find the cycle (stepping-stone tour) Γ_k obtained by adding the nonbasic cell* $\{(m+1, n+1)\}$

*Since $c_{m+1,n+1} = M$, $\{(m+1, n+1)\}$ never belongs to a basis. (This assumes that the problem is primal nondegenerate which can be assured by slightly perturbing the a'_i and b'_j of the problem [9]).

to B_k. Let the cells in the cycle be consecutively numbered as:

$$\Gamma_k = \{(i_0, j_0) = (m+1, n+1)\} \cup \{(i_1, j_1), (i_2, j_2), \ldots (i_{2l-2}, j_{2l-2})\, (i_{2l-1}, j_{2l-1})\}.$$

Define the sets $\Gamma_{1,k}$ (cells with odd parity) and $\Gamma_{2,k}$ (cells with even parity) as follows:

$$\Gamma_{1,k} = \{(i_1, j_1), (i_3, j_3), \ldots (i_{2l-1}, j_{2l-1})\} \text{ and}$$

$$\Gamma_{2,k} = \{(i_0, j_0), (i_2, j_2), \ldots (i_{2l-2}, j_{2l-2})\}.$$

4. Determine $\mu_k = \underset{(i,j)\in\Gamma_{1,k}}{\text{Min}} [x_{ij,k}]$.† let the minimum occur at cell (r, s). If $(r, s) \in \psi$, set $t' = k$ and go to step 8. Otherwise determine the optimum cost Z_{k+1} and solution X_{k+1} as follows:

$$Z_{k+1} = Z_k - \mu_k (u_{m+1,k} + v_{n+1,k}). \tag{42}$$

$$x_{ij,k+1} = \begin{cases} x_{ij,k} - \mu_k & \text{for } (i,j) \in \Gamma_{1,k} \\ x_{ij,k} + \mu_k & \text{for } (i,j) \in \Gamma_{2,k} - \{(m+1, n+1)\} \\ x_{ij,k} & \text{elsewhere.} \end{cases} \tag{43}$$

5. delete the cell (r, s) from B_k. If there are no cells in row r, define $I_r = r$, $J_r = \phi$, $I_s = I' - \{r\}$, $J_s = J'$ and go to (6). Otherwise imagine the remaining basis cells to be connected by horizontal and vertical lines whenever two of the cells lie in the same row or column. Define I_r and J_r to be those lines (rows and columns) that are connected to row r. Set $I_s = I' - I_r$, $J_s = J' - J_r$.**

6. Find ††

$$\Delta = \text{Min} \begin{cases} c_{ij} - u_i - v_j & \text{for } (i,j) \in [I_s \times J_r] \cap \tilde{\Psi} \\ u_i + v_j - c_{ij} & \text{for } (i,j) \in [I_r \times J_s] \cap \Psi \end{cases} \tag{44}$$

Let the minimum occur at cell (e, f). Set $B_{k+1} = B_k + \{(e, f)\}$.

7. Change the dual solutions as follows:

$$u_{i,k+1} = \begin{cases} u_{i,k} - \Delta & \text{for } i \in I_r \\ u_{i,k} & \text{for } i \in I_s, \end{cases} \tag{45}$$

$$v_{j,k+1} = \begin{cases} v_{j,k} + \Delta & \text{for } j \in J_r \\ v_{j,k} & \text{for } j \in J_s. \end{cases} \tag{46}$$

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†This expression for μ is superficially different from the one given in [8, Equation (26)]. This is because all the cells in $\Gamma_{2,k} - \{(i_0, j_0)\} \subset I \times J$ and hence have no upper bounds. The last statement is true since (i) the $(m+1)$st row ($(n+1)$st column) has two cells of Γ (ii) since $\{(i_0, j_0) = (m+1, n+1)\} \in \Gamma_2$ and since alternate cells in Γ belong to different sets, the other cell in row $m+1$ (column $n+1$) belongs to Γ_1. (iii) Thus $\Gamma_2 - \{(i_0, j_0)\} \subset I \times J$. Since these cells do not belong to Ψ they are not bounded from above.

**For a more rigorous definition of these sets see [9]. In Figure 1d, if $(r, s) = (3, 2)$, $I_r = \{3, 4\}$ $J_r = \{1, 3\}$. If $(r, s) = (1, 2)$, $I_r = \{1\}$, $J_r = \{5\}$. If $(r, s) = (2, 4)$ $I_r = \{1, 2, 3, 4\}$, $J_r = \{1, 2, 3, 5\}$.

††This corresponds to Equation (58) in [9]. In the growth path problem, the cell (r, s) to be removed is always such that $x_{rs,k+1} = 0$. (See Step (4) of the algorithm and (43)). Furthermore the sets over which the minimization in (58) of [9] is done can be shown to reduce the sets given in (44).

Set $k=k+1$. Go to (3).

8. The 'corner' points of the growth path have been determined. For any K (such that $K_o < K \leq K_o+N$) to find the optimum solution, find $\delta = K-K_o$, and an index t, such that*

$$\sum_{k=1}^{t-1} \mu_k < \delta \leq \sum_{k=1}^{t} \mu_k \text{ and} \tag{47}$$

$$\lambda = \delta - \sum_{k=1}^{t-1} \mu_k. \tag{48}$$

Then the optimum primal solution X and total cost Z are given by

$$x_{ij} = \begin{cases} x_{ij,\,t} - \lambda & \text{for } (i,j) \in \Gamma_{1,t} \\ x_{ij,\,t} + \lambda & \text{for } (i,j) \in \Gamma_{2,\,t} - \{(m+1, n+1)\} \\ x_{ij,\,t} & \text{elsewhere.} \end{cases} \tag{49}$$

$$Z = Z_t - \lambda(u_{m+1,\,t} + v_{n+1,\,t}). \qquad \text{STOP.} \tag{50}$$

REMARK 1: From (42) and (50) it follows that the cost for any δ such that $0 \leq \delta \leq N$ is given by (where $\delta = K-K_o$)

$$Z = Z_1 - \sum_{k=1}^{t-1} [\mu_k\ (u_{m+1,\,k} + v_{n+1,\,k})] - \lambda(u_{m+1,\,t} + v_{n+1,\,t}), \tag{51}$$

where t and λ are defined as in (47)–(48). Thus the marginal cost of the system (i.e., rate at which the total optimal cost increases when δ is increased) is given by $-(u_{m+1,\,t} + v_{n+1,\,t})$. From [9] it is known that $-(u_{m+1,\,k} + v_{n+1,\,k})$ is a nondecreasing function of k for the operator $\delta \mathbf{R}^{-}_{m+1,\,n+1}$. Consequently, the marginal cost is nondecreasing in δ (and hence in $K = K_o + \delta$).

THEOREM 2: The optimal values for the variables $x_{i,n+1}$ for $(i \epsilon I)$ and $x_{m+1,j}$ (for $j \epsilon J$) as determined by Algorithm 1 are nonincreasing functions of $\delta = K - K_o$.

PROOF: From (47)–(48) it follows that the proof of the theorem would be complete if we can show that it holds for

$$\sum_{k=1}^{t-1} \mu_k \leq \delta \leq \sum_{k=1}^{t} \mu_k \qquad \text{for } \textit{any } t,\ 1 \leq t \leq t'.$$

Defining λ as in (48), it remains to show that it holds for $0 \leq \lambda \leq \mu_t$. (Note that $x_{ij,\,t+1}$ as defined in (43) is the same as that given by (49) for $\lambda = \mu_t$).

From (49), x_{ij} increases only for the cells in $\Gamma_{2,t} - \{(m+1, n+1)\}$. Thus the proof would be complete if we show that no cell $(i,\ n+1)$ for $i \epsilon I$ or $(m+1,\ j)$ for $j \epsilon J$ lies in $\Gamma_{2,\,t}$. To prove this assume the contrary—say, $(p,\ n+1) \in \Gamma_{2,t}$ with $p \epsilon I$. Since the cycle Γ_t has exactly two cells in column $(n+1)$,

*We use the convention $\sum_{i=1}^{0} (\cdot) = 0$.

$(p, n+1)$, and $(m+1, n+1)$ are adjacent cells in the cycle and consequently cannot both belong to $\Gamma_{2,t}$. A similar contradiction would be obtained if we assumed that $(m+1, q) \in \Gamma_{2,t}$ with $q \in J$.

REMARK 2: From (20)–(21) and Theorem 2, it follows that the amounts S_i for $i \in I$ and T_j for $j \in J$ are nondecreasing functions of K. In the production-distribution system where I denotes the set of factories and J denotes the markets, this means that, when the total physical volume is increased from K to K', the optimum soultion to K' can be obtained from that of K by merely *adding* capacities to certain factories and *increasing* the requirements of certain markets (i.e., there is no need to decrease some capacity that has already been added to or some market requirement already increased).

THEOREM 3: Algorithm 1 can be used for finding the optimum solution to (26)–(31) for any value of $K_o \leq K \leq K_o + N$.

PROOF: We need not justify the individual steps in Algorithm 1 here since Phase 1 follows from Theorem 1 and the proofs for Phase 2 appear in [8, 9]. The proof of this theorem would be complete if we can show that

$$\sum_{k=1}^{t'} \mu_k \geq N$$

(see (47)) where t' is defined in step 4 of the algorithm.

From (43), it follows that

$$x_{ij,\,k+1} \geq x_{ij,\,k} - \mu_k, \text{ or}$$

$$x_{ij,\,k+1} - x_{ij,\,k} \geq -\mu_k \qquad \text{for } k = 1, 2, \ldots t' - 1.$$

Adding the above equations for $k = 1$ to $t' - 1$, we get

$$x_{ij,\,t'} - x_{ij,\,1} \geq -\sum_{i=1}^{t'-1} \mu_k. \tag{52}$$

In step (4) of Algorithm 1, $t' = k$ when $(r, s) \in \Psi$ so that $\mu_{t'} = x_{rs,\,t'}$. It will be recalled from step (1) of the algorithm that $x_{rs,\,1} = N$. Thus from (52),

$$\mu_{t'} = x_{rs,\,t'} \geq N - \sum_{k=1}^{t'-1} \mu_k, \qquad \text{or } \sum_{k=1}^{t'} \mu_k \geq N,$$

thus completing the proof.

EXAMPLE 1: Figure 1a shows a transportation problem (32)–(35) with three origins and four destinations. The (i, j)th entry in the tableau denotes the unit cost of shipping from the ith origin to the jth destination. The minimum supplies a_i from the origins and the minimum requirements b_j of the destinations are shown in the rims of this tableau. The unit costs g_i of supplying from the three origins and h_j of supplying to the destinations are also shown in Figure 1a. We note that

$$\sum_{i \in I} a_i = \sum_{j \in J} b_j = 31 = K_o.$$

u_i \ v_j	0	8	-4	3	a_i	g_i
0	9	⑧^{10}	7	6	10	4
2	8	⑩^{0}	6	⑤^{6}	6	2
7	⑦^{5}	⑮^{4}	③^{6}	16	15	1
b_j	5	14	6	6		
h_j	3	1	5	2		

FIGURE 1a

u_i \ v_j	0	8	-4	3	-4	a_i'
0	9	⑧^{10}	7	6	④^{N}	N+10
2	8	⑩^{0}	6	⑤^{6}	-2^{N}	N+6
7	⑦^{5}	⑮^{4}	③^{6}	16	-1^{N}	N+15
-1	-3^{N}	-1^{N}	⑤^{N}	-2^{N}	M	4N
b_j'	N+5	N+14	N+6	N+6	3N	

FIGURE 1b. $(X_1, B_1, u_{i,1}, v_{j,1}), k=1, K=K_o=31$

u_i \ v_j	0	8	-4	3	-4	a_i'
0	9	⑧$^{10+\lambda}$	7	6	④$^{N-\lambda}$	N+10
2	8	⑩^{0}	6	⑤^{6}	-2^{N}	N+6
7	⑦^{5}	⑮$^{4-\lambda}$	③$^{6+\lambda}$	16	-1^{N}	N+15
-1	-3^{N}	-1^{N}	⑤$^{N-\lambda}$	-2^{N}	M	4N-λ
b_j'	N+5	N+14	N+6	N+6	3N-λ	

FIGURE 1c. X for $31 \leq K \leq 35$, $\lambda = K-31$

u_i \ v_j	0	8	-4	3	-4	a_i'
0	9	⑧^{14}	7	6	④$^{N-4}$	N+10
2	8	⑩^{0}	6	⑤^{6}	-2^{N}	N+6
7	⑦^{5}	⑮^{0}	③^{10}	16	-1^{N}	N+15
-1	-3^{N}	-1^{N}	⑤$^{N-4}$	-2^{N}	M	4N-4
b_j'	N+5	N+14	N+6	N+6	3N-4	

FIGURE 1d. $(X_2, B_1, u_{i,1}, v_{j,1}), K=35$

u_i \ v_j	4	8	0	3	-4	a_i'
0	9	⑧^{14}	7	6	④$^{N-4}$	N+10
2	8	⑩^{0}	6	⑤^{6}	-2^{N}	N+6
3	⑦^{5}	15	③^{10}	16	①^{N}	N+15
-5	-3^{N}	-1^{N}	⑤$^{N-4}$	-2^{N}	M	4N-4
b_j'	N+5	N+14	N+6	N+6	3N-4	

FIGURE 1e. $(X_2, B_2, u_{1,2}, v_{j,2}), k=2, K=35$

u_i \ v_j	4	8	0	3	-4	a_i'
0	9	⑧^{14}	7	6	④$^{N-4}$	N+10
2	8	⑩^{0}	6	⑤^{6}	-2^{N}	N+6
3	⑦^{5}	15	③$^{10+\lambda}$	16	①$^{N-\lambda}$	N+15
-5	-3^{N}	-1^{N}	⑤$^{N-4-\lambda}$	-2^{N}	M	4N-4-λ
b_j'	N+5	N+14	N+6	N+6	3N-4-λ	

FIGURE 1f. X for $35 \leq K \leq 31+N$, $\lambda = K-35$

FIGURE 1. Optimum growth path for $K \geq K_o = \sum_{i \in I} a_i = \sum_{j \in J} b_j$ (Example 1)

By assumption (A2), we are interested in studying the growth path of this system as K is increased from $K_o = 31$.

Using the MODI method (primal transportation algorithm) we obtain the optimum solution shown in Figure 1a to the Problem (32)–(35). The circled cells denote the optimal basis and amounts x_{ij} are shown in the northeast corner of the cells. ($x_{ij} = 0$ if the northeast corner is left blank.) The reader may verify that this solution is feasible, i.e., it satisfies (33)–(35). The solution is also optimal since the

dual variables u_i and v_j (shown on the rims of the tableau of Figure 1a) satisfy the dual feasibility conditions $c_{ij} = u_i + v_j$ for $(i, j) \in B$ and $c_{ij} \geq u_i + v_j$ for $(i, j) \notin B$. The value for

$$\sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}$$

for this solution is \$223. From (24) and (32), $Z = \Sigma a_i g_i + \Sigma b_j h_j + 223 = (10 \times 4) + (6 \times 2) + (15 \times 1) + (5 \times 3) + (14 \times 1) + (6 \times 5) + (6 \times 2) + 223 = \$361$.

Figure 1b shows the tableau for the problem (26)–(31). From step (1) of Algorithm 1 the optimum solution $\{x_{ij}\}$ to this problem is the same as that of Figure 1a for $(i, j) \in I \times J$. We set $x_{i,5} = N$ for $i = 1, 2, 3$ and $x_{4,j} = N$ for $j = 1, 2, 3, 4$, and $x_{45} = 0$. Using (36),

$$v_5 = \max [-(0 + 4), -(2 + 2), -(7 + 1)] = -4.$$

We arbitrarily break the tie by taking $p = 1$ so that (1, 5) belongs to the new basis B'. Similarly from (37),

$$u_4 = \max [-(0+3), -(8+1), -(-4+5), -(3+2)] = -1$$

so that (4, 3) also belongs to B'. Figure 1b thus gives the basic optimum solution to (26)–(31) for $K = K_o = 31$ (the optimality conditions (38)–(40) may be verified).

To trace the growth path, we apply the operator $\delta \mathbf{R}_{4,5}^{-}$ using Phase 2 of Algorithm 1. Adding (4, 5) to the basis B_1 we get the cycle, $\Gamma_1 = \{(4, 5), (4, 3), (3, 3), (3, 2), (1, 2), (1, 5)\}$ (Fig. 1c.) so that $\Gamma_{1,1} = \{(4, 3), (3, 2), (1, 5)\}$ and $\Gamma_{2,1} = \{(4, 5), (3, 3), (1, 2)\}$. In step (4), $\mu_1 = x_{32} = 4$ and $(r, s) = (3, 2)$. Consequently $Z_2 = Z_1 - \mu_1(u_{4,1} + v_{5,1}) = 361 - 4(-1-4) = 381$. The solution X_2 obtained from (43) is shown in Figure 1d. The reader may verify that the value of the objective function (26) associated with this solution is $Z_2 = 381$. From step (8) of the algorithm the optimum solution for $31 \leq K \leq 35$ is given by Figure 1c* after setting $\lambda = K - 31$, with optimal cost $Z = Z_1 - \lambda(u_{4,1} + v_{5,1})$ i.e. $Z = 361 + 5\lambda$, i.e., marginal cost is \$5/unit. We note that this is despite the fact that increasing the capacity of origin 1 and destination 3 costs $g_1 + h_3 = \$9$/unit. In other words, if the costs g_1 and h_3 were zero, a simultaneous *increase* of origin 1 and destination 3 actually *decreases* the total cost at the rate of \$4/unit. This phenomena, known as the "transportation paradox" [2, 11] occurs since the alteration of shipments shown in Figure 1c results in increasing x_{12} and x_{33} at a total of \$11/unit whereas the decrease of (3, 2) results in \$15/unit cost reduction so that the net cost reduces at the rate of \$4/unit.

To find the growth path for $K > 35$, we proceed to step (5) of Algorithm 1. After deleting $(r, s) = (3, 2)$ from B_1 we find $I_r = \{3, 4\}$ and $J_r = \{1, 3\}$ so that $I_s = \{1, 2\}$ and $J_s = \{2, 4, 5\}$. From (41) we obtain $\Psi = \{(1, 5), (2, 5), (3, 5), (4, 1), (4, 2), (4, 3), (4, 4)\}$ so that $[I_s \times J_r] \cap \tilde{\Psi} = \{(1, 1), (1, 3), (2, 1), (2, 3)\}$ and $[I_r \times J_s] \cap \Psi = \{(3, 5), (4, 2), (4, 4)\}$. From (44) we obtain $\Delta = \text{Min}[(9-0-0), (7-0+4), (8-2-0), (6-2+4); (7-4+1), (-1+8+1), (-1+3+2)] = 4$. We break the tie arbitrarily by choosing $(e, f) = (3.5)$ so that B_2 is a shown in Figure 1e with the duals changed as per (45)–(46).† We note that this solution is also optimal i.e., it satisfies (38)–(40).

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*The reader may verify that Figure 1d is the same as Figure 1c for $K = 35$, i.e., for $\lambda = \mu_1 = 4$.

†If we choose the alternate cell (4, 4) to enter the basis, the number of iterations is greater and we obtain an *alternate optimum* growth path (i.e., different X, but the same optimal cost Z for values of $K > 35$, from the one obtained in our example).

For iteration 2, we obtain the cycle $\Gamma_2 = \{(4, 5), (4, 3), (3, 3), (3, 5)\}$ (see Fig. 1f) with $\Gamma_{1,2} = \{(4, 3), (3, 5)\}$ and $\Gamma_{2,2} = \{(4, 5), (3, 3)\}$. Thus $\mu_2 = \text{Min}\,[N-4, N] = N-4$ so that $(r, s) = (4, 3)$. Since $(r, s) \in \Psi$, we set $t' = 2$. The optimum solution for values of K such that $35 \leq K \leq 31+N$ is shown in Figure 1f (with $\lambda = K-35$) with optimal cost $Z = Z_2 - \lambda(u_{4,2} + v_{5,2}) = 381 + 9\lambda$. The marginal cost is $-(u_{4,2} + v_{5,2}) = 9$ greater than the marginal cost of 5 in the previous iteration, thus verifying Remark 1. The reader may also verify the statements of Theorems 2 and 3 for this example.

The final solution (in Fig. 1f) for $K \geq 35$ (since N may be chosen to be arbitrarily large, no upper bound on K need be imposed), has the property that cycle $\Gamma_{t'}$ has exactly four cells with only one cell in the $I \times J$ area. This property holds, in general, since if there were more cells in the cycle, then there will be at least one cell (p, q) of the cycle in the $I \times J$ area such that it belongs to $\Gamma_{1,t'}$. From (49) then, as K is increased arbitrarily large x_{pq} would, at some point, become negative thus violating (29).†

The final solution also has the property that the marginal cost $c_{33} + g_3 + h_3 = 9 \leq c_{ij} + g_i + h_j$ for $i \in I$ and $j \in J$. Again such a property holds in general, and can be proved by assuming the contrary, i.e., there exists $(p, q) \in I \times J$ such that $\theta = c_{pq} + g_p + h_q < \alpha$, the marginal cost at the final iteration. But this leads to the contradiction that a cheaper solution can be constructed for values of

$$K > K_0 + \sum_{k=1}^{t'-1} \mu_k$$

by increasing the shipment along (p, q) rather than the cell given by the optimum solution.

Growth Paths When $K < K_a$ and/or $K < K_b$

We showed earlier that the growth path problem as formulated in (3)–(9) has a feasible solution only if (12) and (14) are satisfied, i.e., when K, the total volume required to be handled is at least equal to K_a (K_b), the total of the minimum supplies to be made from the origins (to the destinations). We now consider logistics systems for which the total volumes handled are not sufficient to meet all the minimum requirements. Thus there is a total infeasibility $G = K_a - K$ at the origins and/or $H = K_b - K$ at the destinations. As the total volume is increased from K to $K + \delta$, we will require the new solution to be such that infeasibility (if any) *should* also be reduced by the same extent δ. Such a requirement is consistent with the spirit of imposing minimum requirements. However, other formulations are possible (see section 3).

We now assume that

(A3) $$K_a = \sum_{i \in I} a_i \leq \sum_{j \in J} b_j = K_b.$$

This involves no loss of generality, since otherwise, we can interchange the roles of origins and destinations. We consider below two possible ranges for K:

Case 1: $0 \leq K \leq K_a$

Case 2: $K_a \leq K \leq K_b$.

†In [8] it is shown that the application of a rim operator always preserves primal feasibility conditions, such as (29).

CASE 1: $0 \leq K \leq K_a$

In this case, from (A3) it follows that $K \leq K_b$ also. The formulation of the constraints (3)–(7) still hold. However (8) and (9) get modified as

$$S_i \leq 0 \qquad \text{for } i \epsilon I \tag{53}$$

$$T_j \leq 0 \qquad \text{for } j \epsilon J. \tag{54}$$

The values $-S_i(-T_j)$ denotes the amount by which the minimum requirement at origin i (destination j) is not satisfied. Constraints (53)–(54) are necessary in order to make the formulation realistic. Since the system wishes to maintain total infeasibility at a minimal level, we do not allow some origin to be supplying a surplus ($S_i > 0$) when some other origin's minimum requirement is not fully met ($S_p < 0$). We, however, do not impose any priorities as to which requirements are to be first met etc.,*—all we are concerned about is to keep the *total* infeasibility at a minimum for every level of K.

If we define

$$x_{i,n+1} = -S_i,\ a_i' = a_i \text{ and } c_{i,n+1} = -g_i \qquad \text{for } i \epsilon I, \tag{55}$$

$$x_{m+1,j} = -T_j,\ b_j' = b_j \text{ and } c_{m+1,j} = -h_j \qquad \text{for } j \epsilon J, \tag{56}$$

$$a_{m+1} = \sum_{j \epsilon J} b_j - K,\ b_{n+1} = \sum_{i \epsilon I} a_i - K, \tag{57}$$

$$c_{m+1,n+1} = M \text{ (a large positive number), and} \tag{58}$$

$$Z_o = \sum_{i \epsilon I} g_i a_i + \sum_{j \epsilon J} h_j b_j. \tag{59}$$

Equations (3)–(5), (7), (11), (13), and (53)–(54) become the formulation (26)–(29) i.e., same as the problem solved before but without the upper bound constraints (30)–(31). Since this is a standard transportation problem the optimum solution for any initial value of K, say, $\hat{K}$ can be obtained readily by solving it by the MODI method.

When K is increased from $\hat{K}$ (but keeping within the range $\hat{K} \leq K \leq K_a$) by an amount δ, as mentioned earlier, we want the total infeasibility G and H to reduce by δ, so that a_{m+1} becomes $a_{m+1} - \delta$ and b_{n+1} becomes $b_{n+1} - \delta$. Thus again the operator $\delta \mathbf{R}^{-}_{m+1,n+1}$ will effect the necessary changes in the optimal solution (for $0 \leq \delta \leq K_a - \hat{K}$). Thus phase 2 of Algorithm 1 can be used for this purpose with the following minor modifications: Define X to be the optimum solution to (26)–(29) with the costs and rims defined as in (55)–(59) with $K = \hat{K}$. Let B be the optimal basis, Z be the optimal cost and u_i (for $i \epsilon I'$) and v_j (for $j \epsilon J'$) to be the optimal dual variables. We define $\Psi = \phi$ since in our formulation there

* Priorities could be assigned by appropriately redefining the costs g_i and h_j. For instance, if we require that the third origin's and then the seventh origin's minimum requirement should be met before any others, we can redefine $g_3 = g_3 - M^2$ and $g_7 = g_7 - M$ and leave the others unchanged (M, as usual, is a large positive number). The objective function, accordingly, is redefined as $Z' = Z + M^2 a_3 + M a_7$.

are no variables bounded from above. The termination condition in step (4) should be changed from "if $(r, s) \in \Psi$ set $t' = k$ and go to (8)" to "if $\sum_{i=1}^{k} \mu_i = K_a - \hat{K}$, set $t' = k$ and go to (8)". Similarly the sentence in (8) starting "For any K . . ." should be changed to "For any K (such that $\hat{K} \leq K \leq K_a$) to find the optimum solution, define $\delta = K - \hat{K}$ and an index. . . ." We illustrate the determination of growth path for the case $0 \leq \hat{K} \leq K \leq K_a$ with an example below.

EXAMPLE 2: We consider the same example as before with a_i's somewhat changed as shown in Figure 2a. We note that $K_a = \sum_{i=1}^{3} a_i = 24$ and $K_b = \sum_{j=1}^{4} b_j = 31$ so that (A3) is satisfied. We wish to study the behavior of the optimum solution when K is increased from $\hat{K} = 18$ (say) to $K_a = 24$.

To obtain the initial solution X corresponding to $K = \hat{K} = 18$, we note from (57) that $a_4 = 31 - 18 = 13$ and $b_5 = 24 - 18 = 6$. Solving this problem using the MODI method we get the optimal basis B_1, the solution $\{x_{ij,1}\}$, and the duals $\{u_{i,1}\}$ and $\{v_{j,1}\}$ as shown in Figure 2a. The optimal value of the objective

(26) $Z_1 = Z_o + \sum_{i \in I'} \sum_{j \in J'} c_{ij} x_{ij}$. From (59), $Z_o = \sum_{i=1}^{3} g_i a_i + \sum_{j=1}^{4} h_j b_j = [(4 \times 5) + (2 \times 4) + (1 \times 15)] + [(3 \times 5) + (1 \times 14) + (5 \times 6) + (2 \times 6)] = 114$. From Figure 2(a) $\sum_{i \in I'} \sum_{j \in J'} c_{ij} x_{ij} = 68$ so that $Z_1 = 182$. We note that the optimal solution uses supplies below the minimum requirement for origins 1 and 3 to an extent of 2 and 4 units, respectively. It is interesting that only the requirement of destination 2 is not satisfied, despite the fact its unit cost ($h_2 = 1$) is the smallest among all destinations. (This happens because of the particularly high transportation costs to destination 2.)

To trace the growth path, we apply the operator $\delta \mathbf{R}_{45}^{-}$ using phase 2 of Algorithm 1. At iteration 1, we add (4, 5) to the basis B_1 and obtain the cycle shown in Figure 2b, $\Gamma_1 = \{(4, 5), (4, 2), (1, 2), (1, 5)\}$ with $\Gamma_{1,1} = \{(4, 2), (1, 5)\}$ and $\Gamma_{1,2} = \{(4, 5), (1, 2)\}$. In step (4) we find $\mu = \text{Min}\,(x_{42}, x_{15}) = x_{15} = 2$ with $(r, s) = (1, 5)$. Thus $Z_2 = Z_1 - \mu_1(u_{4,1} + v_{5,1}) = 182 - 2(-9 - 4) = 208$. The solution X_2 obtained using (43) is shown in Figure 2c. Figure 2b shows the optimum solution for $18 \leq K \leq 20$, with λ set equal to $K - 18$. The optimal cost for this solution is $Z = Z_1 - \lambda(u_{4,1} + v_{5,1}) = 182 + 13\lambda$, i.e., the marginal cost is \$13/unit.

To find the optimal solutions for $K > 20$ we need to 'shift' the basis cell $(r, s) = (1, 5)$. We find $I_r = \{1, 2, 4\}$, $J_r = \{2, 4\}$, $I_s = \{3\}$ and $J_s = \{1, 3, 5\}$. Since $\Psi = \phi$, Equation (44) becomes $\Delta = \text{Min}\,\{(c_{ij} - u_i - v_j)\}$ for $(i, j) \in [I_s \times J_r]$. Thus $\Delta = \text{Min}\,\{(c_{32} - u_{3,1} - v_{2,1}), (c_{34} - u_{3,1} - v_{4,1})\} = \text{Min}\,\{4, 7\} = 4$ with $(e, f) = (3, 2)$. Consequently $B_2 = B_1 - \{(1, 5)\} + \{(3, 2)\}$ and the duals u_i and v_j get modified (see (45)–(46)) as shown in Figure 2d.

For iteration 2, we obtain the cycle $\Gamma_2 = \{(4, 5), (4, 2), (3, 2), (3, 5)\}$ shown in Figure 2e. $\mu_2 = 4 = x_{34}$. Since $\sum_{k=1}^{2} \mu_k = 6 = K_a - \hat{K}$, iteration 2 is also the last iteration. Figure 2e shows the optimum solution for $20 \leq K \leq 24$ with cost $Z = Z_2 - \lambda(u_{4,2} + v_{5,2}) = 208 + 17\lambda$ (where $\lambda = K - 20$). The reader may verify that Figure 2f is the same as Figure 2e for $\lambda = 4$, i.e., when $K = K_a = 24$. The optimal cost $Z = 208 + (17 \times 4) = 276$ may be directly verified to be so from Figure 2f. We note that the marginal cost of the system has increased from 13 to 17 verifying Remark 1. The reader may also verify that Theorem 2 holds for this example.

$u_i \backslash v_j$	4	8	0	6	-4	a_i'
0	9	(8)[1]	7	(6)[2]	(-4)[2]	5
-1	8	10	6	(5)[4]	-2	4
3	(7)[5]	15	(3)[6]	16	(-1)[4]	15
-9	-3	(-1)[13]	-5	-2	M	13
b_j'	5	14	6	6	6	

FIGURE 2a. $(X_1, B_1, u_{i,1}, v_{j,1}), k=1, K=18$

$u_i \backslash v_j$	4	8	0	6	-4	a_i'
0	9	(8)[1+λ]	7	(6)[2]	(-4)[2-λ]	5
-1	8	10	6	(5)[4]	-2	4
3	(7)[5]	15	(3)[6]	16	(-1)[4]	15
-9	-3	(-1)[13-λ]	-5	-2	M	13-λ
b_j'	5	14	6	6	6-λ	

FIGURE 2b. X for $18 \leq K \leq 20, \lambda = K - 18$

$u_i \backslash v_j$	4	8	0	6	-4	a_i'
0	9	(8)[3]	7	(6)[2]	(-4)[0]	5
-1	8	10	6	(5)[4]	-2	4
3	(7)[5]	15	(3)[6]	16	(-1)[4]	15
-9	-3	(-1)[11]	-5	-2	M	11
b_j'	5	14	6	6	4	

FIGURE 2c. $(X_2, B_1, u_{i,1}, v_{j,1}), K=20$

$u_i \backslash v_j$	4	12	0	10	-4	a_i'
-4	9	(8)[3]	7	(6)[2]	-4	5
-5	8	10	6	(5)[4]	-2	4
3	(7)[5]	(15)[0]	(3)[6]	16	(-1)[4]	15
-13	-3	(-1)[11]	-5	-2	M	11
b_j'	5	14	6	6	4	

FIGURE 2d. $(X_2, B_2, u_{i,2}, v_{j,2}), k=2, K=20$

$u_i \backslash v_j$	4	12	0	10	-4	a_i'
- 4	9	(8)[3]	7	(6)[2]	-4	5
- 5	8	10	6	(5)[4]	-2	4
3	(7)[5]	(15)[λ]	(3)[6]	16	(-1)[4-λ]	15
-13	-3	(-1)[11-λ]	-5	-2	M	11-λ
b_j'	5	14	6	6	4-λ	

FIGURE 2e. X for $20 \leq K \leq 24$

$u_i \backslash v_j$	4	12	0	10	-4	a_i'
- 4	9	(8)[3]	7	(6)[2]	-4	5
- 5	8	10	6	(5)[4]	-2	4
3	(7)[5]	(15)[4]	(3)[6]	16	(-1)[0]	15
-13	-3	(-1)[7]	-5	-2	M	7
b_j'	5	14	6	6	0	

FIGURE 2f. X for $K = K_a = 24$

FIGURE 2. Optimum growth path for $K \leq K \leq K_a$ (Example 2)

CASE 2: $K_a \leq K \leq K_b$

Since $K \geq K_a$, all the minimum requirements with respect to origins can and hence should be met. Thus $S_i \geq 0$ still holds. However, as in Case 1, since $K \leq K_b$, the minimum requirements cannot be met for all destinations so that constraints $T_j \leq 0$ apply. Accordingly, the formulation for this case is a combination of the two formulations already discussed. Consequently, if we define

(60) $$x_{i,n+1}=N-S_i,\ a_i'=a_i+N \text{ and } c_{i,n+1}=-g_i \qquad \text{for } i\epsilon I,$$

(61) $$x_{m+1,j}=-T_j,\ b_j'=b_j \text{ and } c_{m+1,j}=-h_j \qquad \text{for } j\epsilon J,$$

(62) $$a_{m+1}=\sum_{j\epsilon J} b_j-K,\ b_{n+1}=mN+\sum_{i\epsilon I} a_i-K,$$

(63) $$c_{m+1,n+1}=M, \text{ and}$$

(64) $$Z_o=\sum_{i\epsilon I} g_i\ (N+a_i)+\sum_{j\epsilon J} h_j b_j,$$

then this problem becomes the formulation (26)–(30). Thus the variables that are bounded from above belong to the set

(65) $$\Psi=\{(i, n+1) \qquad \text{for } i\epsilon I\}.$$

To trace the growth path, operator $\delta\mathbf{R}^{-}_{m+1,n+1}$ again becomes applicable with the following minor changes in phase 2 of Algorithm 1. The termination condition in step (4) should be changed to "if $\sum_{i=1}^{k}\mu_i=K_b-\hat{K}$, set $t'=k$ and go to 8." Similarly the sentence in (8) starting "For any K . . ." should be changed to "For any K (such that $\hat{K}\leq K\leq K_b$) to find the optimum solution, define $\delta=K-\hat{K}$ and an index . . . ." Note that $\hat{K}$ is the initial value of K from which we wish to have the growth path. Without much loss of generality we can assume $\hat{K}=K_a$ (if $\hat{K}>K_a$ this would only mean that we will be tracing the growth path for the range K_a to $\hat{K}$ unnecessarily). An initial solution to this problem can be obtained by using Case 1 (where the cells $(i, n+1)$ are *not* bounded from above). Let us assume that this basic optimum solution has only one basic cell in the last column* (say $(r, n+1)\ \epsilon\ B$). Then an optimal basis for Case 2 is given by $B'=B-\{(r, n+1)\}+\{(p, n+1)\}$ (where the index p (and v_{n+1}) are defined in (36)) with $x_{i,n+1}$ set equal to N for $i\epsilon I$. We illustrate Case 2 with an example below.

EXAMPLE 3: We take the data for this example to be the same as that of Example 2. Taking $\hat{K}=K_a=24$, we want to trace the growth path for $24\leq K\leq 31=K_b$. A basic optimal solution for $K=24$ is provided in Figure 2f where (3, 5) is the only basic cell in the last column. Applying (36),

$$v_5=\max\{-(u_1+g_1),-(u_2+g_2),-(u_3+g_3)\}=\max\{0,3,-4\}=3$$

so that $B'=B-\{(3, 5)\}+\{(2, 5)\}$ is the new optimal basis with $x_{i,5}=N$ for $i=1, 2, 3$ as shown in Figure 3a. (The rims a_i and b_j are also modified as per (60)–(62)). The cost of this solution can be verified from (26) and (64) to be $Z_1=276$ (same as that of Figure 2f). To trace the growth path, we add (4, 5) to the basis $B_1=B'$ and obtain the cycle shown in Figure 3b so that $\mu_1=2$ with $(r, s)=(1, 4)$. The solution X_2 is shown in Figure 3c with $Z_2=Z_1-\mu_1(u_{4,1}+v_{5,1})=276-2(-13+3)=296$. The optimum solution for $24\leq K\leq 26$ is shown in Figure 3b with $Z=276+10\lambda$. Deleting $(r, s)=(1, 4)$ from B_1, we

*This condition can be ensured by perturbing the minimum requirements a_i to $a_i+\epsilon$ where ϵ is a negligibly small but positive number [9]. Alternatively, the zero cells of a basis would have to be shifted (Steps (5)–(7) of Algorithm 1).

u_i \ v_j	4	12	0	10	3	a_i
-4	9	(8)^3	7	(6)^2	-4^N	N+5
-5	8	10	6	(5)^4	(-2)^N	N+4
3	(7)^5	(15)^4	(3)^6	16	-1^N	N+15
-13	-3	(-1)^7	-5	-2	M	7
b_j	5	14	6	6	3N	

FIGURE 3a. $(X_1, B_1, u_{i,1}, v_{j,1})$, $k=1$, $K=24$

u_i \ v_j	4	12	0	10	3	a_i
-4	9	(8)$^{3+\lambda}$	7	(6)$^{2-\lambda}$	-4^N	N+5
-5	8	10	6	(5)$^{4+\lambda}$	(-2)$^{N-\lambda}$	N+4
3	(7)^5	(15)^4	(3)^6	16	-1^N	N+15
-13	-3	(-1)$^{7-\lambda}$	-5	-2	M	7-λ
b_j	5	14	6	6	3N-λ	

FIGURE 3b. X for $24 \leq K \leq 26$, $\lambda = K-24$

u_i \ v_j	4	12	0	10	3	a_i
-4	9	(8)^5	7	(6)^0	-4^N	N+5
-5	8	10	6	(5)^6	(-2)$^{N-2}$	N+4
3	(7)^5	(15)^4	(3)^6	16	-1^N	N+15
-13	-3	(-1)^5	-5	-2	M	5
b_j	5	14	6	6	3N-2	

FIGURE 3c. $(X_2, B_1, u_{i,1}, v_{j,1})$, $K=26$

u_i \ v_j	7	15	3	10	3	a_i
-7	9	(8)^5	7	6	(-4)^N	N+5
-5	8	10	6	(5)^6	(-2)$^{N-2}$	N+4
0	(7)^5	(15)^4	(3)^6	16	-1^N	N+15
-16	-3	(-1)^5	-5	-2	M	5
b_j	5	14	6	6	3N-2	

FIGURE 3d. $(X_2, B_2, u_{i,2}, v_{j,2})$, $k=2$, $K=26$

u_i \ v_j	7	15	3	10	3	a_i
-7	9	(8)$^{5+\lambda}$	7	6	(-4)$^{N-\lambda}$	N+5
-5	8	10	6	(5)^6	(-2)$^{N-2}$	N+4
0	(7)^5	(15)^4	(3)^6	16	-1^N	N+15
-16	-3	(-1)$^{5-\lambda}$	-5	-2	M	5-λ
b_j	5	14	6	6	3N-2-λ	

FIGURE 3e. X for $26 \leq K \leq 31$, $\lambda = K-26$

u_i \ v_j	7	15	3	10	3	a_i
-7	9	(8)^{10}	7	6	(-4)$^{N-5}$	N+5
-5	8	10	6	(5)^6	(-2)$^{N-2}$	N+4
0	(7)^5	(15)^4	(3)^6	16	-1^N	N+15
-16	-3	(-1)^0	-5	-2	M	0
b_j	5	14	6	6	3N-7	

FIGURE 3f. X for $K=K_b=31$

FIGURE 3. Optimum growth path for $K_b \leq K \leq K_u$ (Example 3)

obtain $I_r=\{1, 3, 4\}$, $J_r=\{1, 2, 3\}$, $I_s=\{2\}$, and $J_s=\{4, 5\}$. From (65) $\Psi=\{(1, 5), (2, 5), (3, 5)\}$ so that $[I_s \times J_r] \cap \tilde{\Psi}=\{(2, 1), (2, 2), (2, 3)\}$ and $[I_r \times J_s] \cap \Psi=\{(1, 5), (3, 5)\}$. From (44) it then follows that $\Delta=3$ with $(e, f)=(1, 5)$.* Figure 3d shows the new basis and duals. Figure 3e shows the optimal X for $26 \leq K \leq 31$ with optimal cost $Z=Z_2+13\lambda$. Again the marginal cost increases from 10 to 13 verifying Remark 1. Figure 3f corresponds to the optimum solution for $K=K_b=31$ with $Z=296+(13\times 5)=361$ (same as the cost of the solution in Figures 1a–1b). We note that the optimum solution of Figure 1a corresponds to the solution of Figure 3f.†

* An alternate optimal growth path would be obtained by choosing $(e, f)=(2, 2)$.

†The cell (2, 2) in Fig. 3f corresponds to an alternate optimum $c_{22}-u_2-v_2=0$. Thus removing (2, 5) and adding (2, 2) will still constitute an optimal basis for Figure 3f. The a_i's in Figure 1a correspond to a_i+S_i in Figure 3f. Also the dual solutions of Figure 1a can be obtained by adding 7 to the u_i's and subtracting 7 from the v_j's of Figure 3f.

3. EXTENSIONS AND GENERALIZATIONS

3.1. Upper Bounded Growth Path Model

The theory and algorithms of section 2 can easily be extended (as in [8, 9]) to the case when constraints (7) are replaced by $0 \leq x_{ij} \leq U_{ij}$, where U_{ij} denotes the maximum amount that can be shipped via route (i, j). While Theorems 1 and 2 will still hold, Theorem 3 may be false since we may not be able to find a growth path for all K in the interval $K_o \leq K \leq K_o + N$, because the U_{ij} may limit the maximum value of K. Naturally the details of proofs of theorems and the steps in Phase II of Algorithm 2 must be suitably modified.

Logistics problems involving the shipment of men and supplies during a limited period of time may require the use of the upper bounded model. Here the upper bounds correspond to capacity limitations of ships or airplanes, loading and unloading facilities, etc.

3.2. Convex Costs of Supplying from Origins (to Destinations)

The methodology of the section 2 can be extended to the case when the costs of supplying $g(S_i)$ and $h(T_j)$ are not necessarily linear, but can be approximated by a convex piecewise linear function. For instance, let us consider the case when (A2) holds, viz., $K \geq K_o = \Sigma a_i = \Sigma b_j$. If $a_1 = 20$ and costs of adding an additional unit of supply is \$4/unit for $0 \leq S_1 \leq 10$, \$5 per unit fo $10 < S_1 \leq 25$, and \$6/unit for $S_1 > 25$. Then corresponding to the first origin we may define three fictitious origins ($i = 1, 2, 3$, say) with same transportation costs but with $c_{1,n+1} = -4$, $c_{2,n+1} = -5$ and $c_{3,n+1} = -6$. In any feasible solution let $S_{1,1}, S_{1,2}, S_{1,3}$ denote the surpluses supplied in the ranges 0–10, 10–25, > 25, respectively, so that $S_1 = S_{1,1} + S_{1,2} + S_{1,3}$. Then defining $x_{1,n+1} = 10 - S_{1,1}$, $x_{2,n+1} = (25 - 10) - S_{1,2}$, and $x_{3,n+1} = N - S_{1,3}$, we get the upper bounding constraints ($x_{1,n+1} \leq 10$, $x_{2,n+1} \leq 25 - 10 = 15$, and $x_{3,n+1} \leq N$). Because of the convex nature of the costs, the optimum solution has the property that $S_{1,2} > 0$ (i.e., $x_{2,n+1} < 15$) only if $S_{1,1} = 10$ (i.e., $x_{1,n+1} = 0$); similarly $S_{1,3} > 0$ (i.e., $x_{3,n+1} < N$) only if both $S_{1,1} = 10$ and $s_{1,2} = 15$.*

3.3. Growth Paths with Prespecified Market Growth Rates

Consider a system of factories and markets with supplies a_i (for $i \epsilon I$) and requirements b_j (for $j \epsilon J$) respectively $\left(\text{with} \sum_{i \epsilon I} a_i = \sum_{j \epsilon J} b_j\right)$. The market requirements are expected to increase at the rate of β_j units/month (say). The problem may then be to decide the production levels of the factories for the next L months (say). Denoting by δ the time periods ($0 \leq \delta \leq L$), the market requirements at the δth period would be $b_j + \delta\beta_j$ so that the problem for the δth period can be formulated as

$$\text{Min } Z = \sum_{i \epsilon I} \sum_{j \epsilon J} c_{ij} x_{ij} + \sum_{i \epsilon I} g_i(a_i + S_i) + \sum_{j \epsilon J} h_j(b_j + \delta\beta_j) \tag{66}$$

subject to the constraints

$$\sum_{j \epsilon J} x_{ij} = a_i + S_i \qquad \text{for } i \epsilon I, \tag{67}$$

* A simplified procedure for such problems is possible (without using fictitious origins which increase the size of the problem), guaranteeing global optimality when supply costs are convex and leading to locally optimum solutions otherwise. This technique involving the use of cost operators [8, 9] is presently being investigated by the authors.

(68) $$\sum_{i\epsilon I} x_{ij} = b_j + \delta\beta_j \qquad \text{for } j\epsilon J,$$

(69) $$x_{ij} \geq 0 \qquad \text{for } i\epsilon I \text{ and } j\epsilon J, \text{ and}$$

(70) $$S_i \geq 0 \qquad \text{for } i\epsilon I.$$

Adding N to both sides of (67), and rearranging

(71) $$\sum_{j\epsilon J} x_{ij} + (N - S_i) = a_i + N \qquad \text{for } i\epsilon I.$$

Summing the Equations (71) for $i\epsilon I$ and subtracting from it the summation of (68) over $j\epsilon J$ we get,

(72) $$\sum_{i\epsilon I} (N - S_i) = \left(\sum_{i\epsilon I} a_i - \sum_{j\epsilon J} b_j \right) + mN - \delta \sum_{j\epsilon J} \beta_j.$$

Since we assumed $\sum_{i\epsilon I} a_i = \sum_{j\epsilon J} b_j$, (72) becomes

(73) $$\sum_{i\epsilon I} (N - S_i) = mN - \delta \sum_{j\epsilon J} \beta_j.$$

Similarly (66) can be rewritten as

(74) $$\text{Min } Z = \sum_{i\epsilon I} \sum_{j\epsilon J} c_{ij} x_{ij} - \sum_{i\epsilon I} g_i (N - S_i) + \left[\sum_{i\epsilon I} g_i (a_i + N) + \sum_{j\epsilon J} h_j (b_j + \delta\beta_j) \right].$$

If we define

(75) $$x_{i,n+1} = N - S_i,\ c_{i,n+1} = -g_i \text{ and } a_i' = a_i + N \qquad \text{for } i\epsilon I,$$

(76) $$b_{n+1} = mN \text{ and } \beta_{n+1} = -\sum_{j\epsilon J} \beta_j, \text{ and}$$

(77) $$Z_o = \sum_{i\epsilon I} g_i (a_i + N) + \sum_{j\epsilon J} h_j (b_j + \delta\beta_j),$$

then (74), (71), (68), (73), (69)–(70) become

(78) $$\text{Min } Z = Z_o + \sum_{i\epsilon I} \sum_{j\epsilon J'} c_{ij} x_{ij}$$

subject to the constraints

(79) $$\sum_{j\epsilon J'} x_{ij} = a_i' \qquad \text{for } i\epsilon I,$$

(80) $$\sum_{i\epsilon I} x_{ij} = b_j + \delta\beta_j \qquad \text{for } j\epsilon J',$$

(81) $$x_{ij} \geq 0 \qquad \text{for } i\epsilon I \text{ and } j\epsilon J', \text{ and}$$

(82) $$x_{i,n+1} \leq N \qquad \text{for } i\epsilon I.$$

Let us assume that we have solved the problem (78)–(82) for $\delta = 0$. Then the optimal solution for any other value of δ, i.e., $0 \leq \delta \leq L$ can be obtained by applying the Area Rim Operator $\delta\mathbf{R}_A$[8, 9] with $\alpha_i = 0$ for $i\epsilon I$.* The procedures for applying the operator are discussed in the above references and need not be repeated here.

The network problems of Doulliez and Rao [5, 6] can be put into the form of a growth path transportation problem with prespecified market growth rates in which (70) is replaced by $S_i \leq 0$ for $i\epsilon I$, and after making the upper bounding extensions of section 3.1. Also the determination of μ_k in step 4 of Algorithm 1 should be modified to include the possibility of some of the arcs failing (i.e., a cost c_{ij} changing from 0 to ∞).

3.4. Solving Transportation Problems not in "Standard Form"

Consider the distribution problem

(83) $$\text{Minimize} \sum_{i\epsilon I} \sum_{j\epsilon J} c_{ij}x_{ij}$$

subject to the constraints

(84) $$\sum_{j\epsilon J} x_{ij} \geq a_i \qquad \text{for } i\epsilon I,$$

(85) $$\sum_{i\epsilon I} x_{ij} = b_j \qquad \text{for } j\epsilon J, \text{ and}$$

(86) $$x_{ij} \geq 0 \qquad \text{for } i\epsilon I \text{ and } j\epsilon J,$$

$$\text{where } K_a = \sum_{i\epsilon I} a_i \leq \sum_{j\epsilon J} b_j = K_b.$$

While other procedures [7, p. 250, 3, 8, 9] have been suggested, the growth path algorithms of section 2 provide an alternate procedure for solving this problem. We modify the constraints (85) to

(87) $$\sum_{i\epsilon I} x_{ij} + T_j = b_j \qquad \text{for } j\epsilon J \text{ where } T_j \geq 0.$$

---

* An area rim operator transforms the optimum solution of a problem P to that of a problem P^A whose data are the same as that of P except that supplies a_i are changed to $a_i + \delta\alpha_i$, and requirements b_j changed to $b_j + \delta\beta_j$ (where δ is a parameter and α_i, β_j are given numbers, such that $\sum_{i\epsilon I} \alpha_i = \sum_{j\epsilon J} \beta_j$).

If we set $g_i = 0$ for $i \in I$ and $h_j = 0$ for $j \in J$ and define $\sum_{i \in I} \sum_{j \in J} x_{ij} = K$, then the problem (83), (84), (86), (87) is the same as Case 2 of section 2. Thus starting with the optimum solution for $K = K_a$ and tracing the growth path to $K = K_b$ would yield the solution to (83)–(86).* (The latter value for K would imply that $T_j = 0$ for $j \in J$ so that (85) and (87) are equivalent.)

The distribution problem (83)–(86) with the equality sign in (85) replaced by $\geq$ signs, can be solved as before but increasing the value of K beyond K_b (if necessary) until the marginal cost $-(u_{m+1} + v_{n+1})$ becomes nonnegative.

3.5. A Goal Programming Growth Model

The formulation of the growth model (3)–(9) assumed that the minimum requirements a_i and b_j necessarily have to be met. In certain other contexts, they may be better treated as "goals." Thus we will allow a surplus or deficit in each origin's and destination's minimum requirement.† Let S_i^+ (S_i^-) denote the surplus (deficit) at origin i, and T_j^+ (T_j^-) denote the surplus (deficit) at destination j. Let $\mathbf{S}^+$ ($\mathbf{S}^-$) denote the total surplus (deficit) at the origins and $\mathbf{T}^+$ ($\mathbf{T}^-$) denote the total surplus (deficit) at the destinations. Then denoting by lower case letters the rewards (penalties) associated with the surpluses (deficits)** the formulation becomes:

$$\text{(88)} \quad \text{Min } Z = \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \sum_{i \in I} g_i (a_i + S_i^+ - S_i^-) + \sum_{j \in J} h_j (b_j + T_j^+ - T_j^-) + \sum_{i \in I} (s_i^- S_i^- - s_i^+ S_i^+) + \sum_{j \in J} (t_j^- T_j^- - t_j^+ T_j^+) - \mathbf{s}^+ \mathbf{S}^+ + \mathbf{s}^- \mathbf{S}^- - \mathbf{t}^+ \mathbf{T}^+ + \mathbf{t}^- \mathbf{T}^-.$$

Subject to the constraints:

$$\text{(89)} \quad \sum_{j \in J} x_{ij} = a_i + S_i^+ - S_i^- \qquad \text{for } i \in I,$$

$$\text{(90)} \quad \sum_{i \in I} x_{ij} = b_j + T_j^+ - T_j^- \qquad \text{for } j \in J,$$

$$\text{(91)} \quad \sum_{i \in I} S_i^+ = \mathbf{S}^+,$$

$$\text{(92)} \quad \sum_{i \in I} S_i^- = \mathbf{S}^-,$$

$$\text{(93)} \quad \sum_{j \in J} T_j^+ = \mathbf{T}^+,$$

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* The distribution problem in which the $\geq$ signs in (84) are replaced by equations and the $=$ signs in (85) are replaced by $\geq$ signs can be solved after interchanging the roles of origins and destinations.

† The variables below are defined in such a manner that they take on only nonnegative values.

** We impose a penalty at the rate of $\mathbf{t}^-$ on the total deficit. The penalties t_i are, in addition, imposed to reflect priorities as to which of the requirements are more important. Similarly the penalties (rewards) $\mathbf{s}^-$, s_i^-, $\mathbf{t}^+$, t_i^+, $\mathbf{s}^+$, s_i^+ are interpreted.

(94) $$\sum_{j\epsilon J} T_j^- = \mathbf{T}^-,$$

(95) $$\sum_{i\epsilon I}\sum_{j\epsilon J} x_{ij} = K, \text{ and}$$

(96) $$x_{ij},\ S_i^+,\ S_i^-,\ T_j^+,\ T_j^- \geq 0 \qquad \text{for } i\epsilon I \text{ and } j\epsilon J,\ \mathbf{S}^+,\ \mathbf{S}^-,\ \mathbf{T}^+,\ \mathbf{T}^- \geq 0.$$

To reduce (88)–(96) to a standard capacitated transportation problem we first define $g_i^+ = g_i - s_i^+$, $g_i^- = g_i + s_i^-$, $h_j^+ = h_j - t_j^+$, $h_j^- = h_j + t_j^-$ and $Z_o = \sum_{i\epsilon I} g_i a_i + \sum_{j\epsilon J} h_j b_j$, so that (88) becomes

(97)
$$\text{Min } Z = Z_o + \sum_{i\epsilon I}\sum_{j\epsilon J} c_{ij}x_{ij} + \sum_{i\epsilon I}(g_i^+ S_i^+ + g_i^- S_i^-) + \sum_{j\epsilon J}(h_j^+ T_j^+ + h_j^- T_j^-) - \mathbf{s}^+\mathbf{S}^+ + \mathbf{s}^-\mathbf{S}^- - \mathbf{t}^+\mathbf{T}^+ + \mathbf{t}^-\mathbf{T}^-.$$

By adding N to both sides of (89) and (90) and rearranging we get

(98) $$\sum_{j\epsilon J} x_{ij} + (N - S_i^+) + S_i^- = a_i + N \qquad \text{for } i\epsilon I, \text{ and}$$

(99) $$\sum_{i\epsilon I} x_{ij} + (N - T_j^+) + T_j^- = b_j + N \qquad \text{for } j\epsilon J.$$

Equations (91) and (93) can be rewritten as

(100) $$\sum_{i\epsilon I}(N - S_i^+) + \mathbf{S}^+ = mN, \text{ and}$$

(101) $$\sum_{j\epsilon J}(N - T_j^+) + \mathbf{T}^+ = nN.$$

Similarly (92) and (94) become

(102) $$\sum_{i\epsilon I} S_i^- + (N - \mathbf{S}^-) = N, \text{ and}$$

(103) $$\sum_{j\epsilon J} T_j^- + (N - \mathbf{T}^-) = N.$$

By adding (95), (100), and (102) and subtracting from it the sum of (98) over i, we get

(104) $$\mathbf{S}^+ + (N - \mathbf{S}^-) = K + N - \sum_{i\epsilon I} a_i.$$

Similarly,

(105) $$\mathbf{T}^+ + (N - \mathbf{T}^-) = K + N - \sum_{j\epsilon J} b_j.$$

By defining $Z_o' = Z_o + \sum_{i \epsilon I} g_i^+ N + \sum_{j \epsilon J} h_j^+ N + \mathbf{s}^- N + \mathbf{t}^- N$, (97) becomes

$$\text{(106)} \qquad \text{Min } Z = Z_o' + \sum_{i \epsilon I} \sum_{j \epsilon J} c_{ij} x_{ij} - \sum_{i \epsilon I} g_i^+ (N - S_i^+) + \sum_{i \epsilon I} g_i^- S_i^- - \sum_{j \epsilon J} h_j^+ (N - T_j^+) + \sum_{j \epsilon J} h_j^- T_j^- - \mathbf{s}^+ \mathbf{S}^+ - \mathbf{s}^- (N - \mathbf{S}^-) - \mathbf{t}^+ \mathbf{T}^+ - \mathbf{t}^- (N - \mathbf{T}^-).$$

Since $S_i^+, \mathbf{S}^-, T_j^+, \mathbf{T}^-$ are defined to be nonnegative, we have

$$\text{(107)} \qquad (N - S_i^+) \leqslant N \quad \text{for } i \epsilon I, \ (N - \mathbf{S}^-) \leqslant N, \ (N - T_j^+) \leqslant N \quad \text{for } j \epsilon J, \ (N - \mathbf{T}^-) \leqslant N.$$

The capacitated transportation problem (98)–(107) is shown schematically in Figure 4 as a tableau with $(m+3)$ origins and $(n+3)$ destinations.

After solving this problem for any particular value of K, the optimum solution for some other $K' = K + \delta (\delta \geqslant 0)$ can be obtained by applying the Cell Rim Operator $\delta \mathbf{R}^+_{m+3,\, n+3}$ with suitable modifications made in Phase 2 of Algorithm 1.

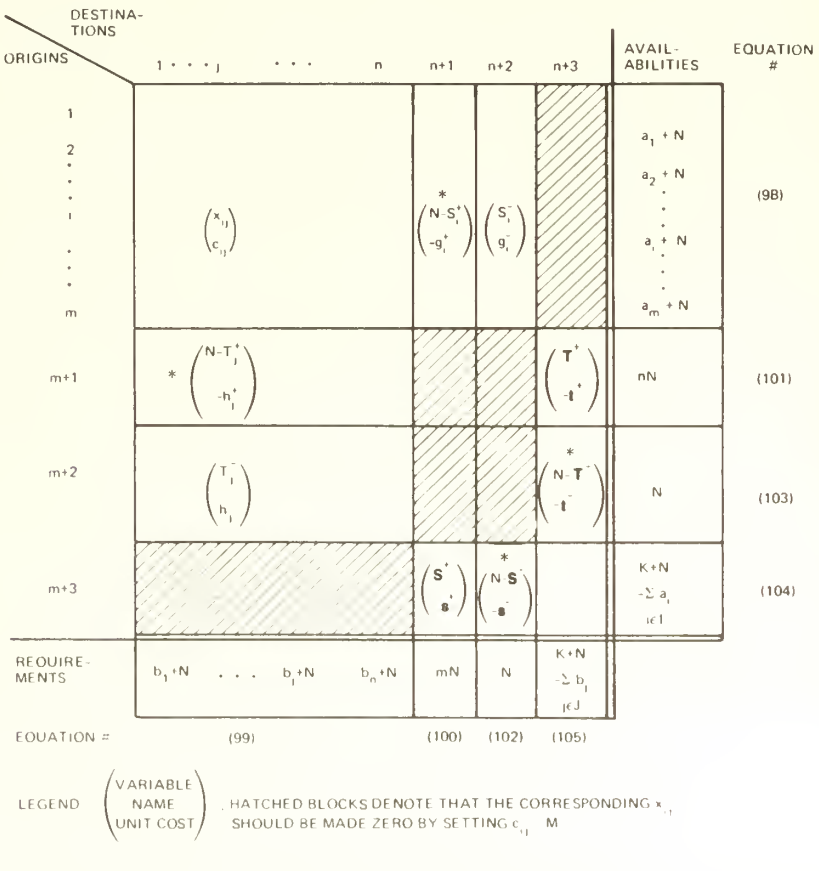

FIGURE 4. Capacitated transportation tableau for the goal programming growth model

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STATIONARY (s, S) POLICIES FOR A FINITE HORIZON*

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ABSTRACT

In the finite-horizon stochastic (s, S) inventory model with periodic review the parameters of the optimal policy generally vary with the length of the horizon. A stationary policy, however, is easier to implement and may be easier to calculate. This paper studies optimal stationary policies for a finite horizon and relates them to optimal policies through their relation to optimal stationary policies for an infinite horizon.

1. INTRODUCTION

Under appropriate conditions on the single-product periodic-review stochastic inventory model with set-up cost an (s, S) policy may be shown to be optimal [11, 12, 13]. In the infinite-horizon problem with stationary parameters a stationary policy is optimal. For the finite-horizon problem, however, stationarity of an optimal policy has been ensured only in the case of guaranteed demand [14]. Thus, when an (s, S) policy is optimal for an n-period problem one generally has a sequence of pairs of parameters $\{(s_t, S_t)\}_{t=1}^{n}$ with (s_t, S_t) determining the ordering rule for the tth period.

This situation has two drawbacks over the stationary one. In the first place, calculating the parameters (s_t, S_t) can be quite difficult. A myopic policy [10] is generally not optimal and a dynamic programming formulation of the problem is used. Unlike the case with a myopic policy this requires calculation of the optimal cost for a vast number of states which won't be entered. Then, for each stage of the recursion one must solve a nonconvex programming problem with some relative minima which aren't global minima. In the second place, implementing the policy can be awkward. When one is using (s, S) policies independently for many products over a long horizon, the number of parameters kept on file by the inventory manager is quite large.

The models in which policies other than (s, S) policies are optimal exhibit similar but greater problems of calculation and implementation. For example, Boylan has shown that under certain conditions what he calls a multiple (s, S) policy is optimal [2, 3]. For such a policy there are characteristically more than one pair of parameters for each period, each pair presenting practical problems comparable to those associated with the single pair (s_t, S_t) of an (s, S) policy.

Finally, there are models with so little structure that nothing is said about them. A policy which is neither easy to calculate nor simple to describe is generally considered not worth reporting. In this situation a person might revise his notion of an optimal policy by reducing the size of the class

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of feasible policies. In practice, the overly ambitious development of a complex model generally necessitates such a revision.

In the present context, then, the inventory manager may be faced with an impractical problem either (1) because the optimal policy is theoretically characterized but too difficult to calculate or to implement, or (2) because it is simply uncharacterized. It appears that a common solution to this dilemma is simply to restrict attention to stationary (s, S) policies and to ignore the fact that such a policy is not optimal in the class of all policies. The characteristics of these policies in a general model is the subject of this paper, which essentially publishes results first reported elsewhere [15]. We consider first their definition and calculation and then investigate questions regarding both bounds on the parameters and convergence of optimal costs and policies as the length of the horizon goes to ∞.

2. THE MODEL

At the beginning of period $t \geq 1$ an amount x_t is on hand. For some $S \epsilon E_1$ and $s \epsilon [-\infty, S]$ if $x_t < s$ an order is placed with immediate delivery bringing inventory up to $y_t = S$, while if $x_t \geq s$ the inventory $y_t = x_t$. If $s = -\infty$ no order is ever placed. Next a random demand ξ_t totally independent of other demands draws on inventory and yields $x_{t+1} = y_t - \xi_t$. It is assumed ξ_t has a probability distribution function $\Phi(\cdot)$, a probability density function $\phi(\cdot)$, and a finite mean μ. As for the modification of Veinott and Wagner [14] the one-period expected holding and penalty cost is expressed by $G(y)$ when y is the level after ordering and the purchase cost by $K(y_t - x_t)$, which is $K > 0$ when $y_t > x_t$ and 0 when $y_t = x_t$. Unlike other treatments we assume only that $G(y)$ is continuous and approaches ∞ as $|y| \to \infty$. Without loss of generality we also assume that $G(y)$ has a positive lower bound. Future costs are discounted by a factor $\alpha \epsilon (0, 1]$. The expected cost for the n-period problem can then be written as

$$f_n(s, S) = \sum_{t=1}^{n} \alpha^{t-1} E[K(y_t - x_t) + G(y_t)].$$

The simple objective of choosing a best (s, S) policy is not completely defined. In the one-period problem, for example, if an unrestricted optimal policy has $y_1 = x_1$ then any (s, S) policy with $s \leq x_1$ is optimal. Similarly, if $y_1 > x_1$ then any (s, S) policy with $s > x_1$ and $S = y_1$ is optimal. To require that some (s, S) policy be optimal for all x_1 implies insisting that for some unrestricted optimal policy $\{x_1 \mid y_1 > x_1\}$ be of the form $(-\infty, s)$. A weak condition for this to be true is that $G(y)$ be quasi-K-convex [11]. For more general forms of $G(y)$ the best (s, S) policy depends on x_1.

As mentioned above, for the unrestricted n-period problem when an (s, S) policy is optimal it is generally not stationary. Thus, even when $G(y)$ is convex the best stationary (s, S) policy not only depends on x_1, but also generally is inferior to the unrestricted optimal policy. This point is even more evident with our weaker assumptions on $G(y)$. For an infinite horizon with $\alpha < 1$ the dependence on x_1 remains.

In any particular problem the initial level is known and the optimal stationary (s, S) policy may then be calculated. To obtain a general solution, however, it is useful to specify an initial level by an assumption which is the same for all problems. For example, it might seem natural to take zero as a common initial level since frequently practical problems do start with no items available and no backlog. It might also be argued, however, that the models in this paper in most cases can't be applied until the inventory process has been observed for a few periods, in order that good assumptions regarding costs and demand can be made. From a mathematical point of view an initial level of zero is unsatisfying

since it is difficult to compare the general cost of the best (s, S) policy with $s \geq 0$ with that of perhaps a different one with $s \leq 0$.

To resolve this difficulty we shall adopt the convention of assuming that the initial level is S and thus set it to one of the parameters with respect to which the minimization is performed. If (s^*, S^*) is an optimal policy under this assumption, then it is also an optimal stationary (s, S) policy for all initial levels $x < s^*$ for which it is optimal to order in the first period. If $0 < s^*$, for example, and an optimal stationary (s, S) policy for an initial level of zero is $(s(0), S(0))$ where $0 < s(0)$, then it is optimal to take $(s(0), S(0)) = (s^*, S^*)$. On the other hand, note that (s^*, S^*) is certain to be optimal for an initial level $x = S^*$ only if $S(S^*) = S^*$.

Adapting the notation of Greenberg [6, 7] to the present needs let

$$F_t(x|s, S) = P[x_{t+1} \leq x \text{ when } (s, S) \text{ is followed each period}],$$

$$r_t(S - s) = F_t(s - |s, S)$$

for s finite and $t \geq 0$, where $x_1 = S$ by assumption. Note that, as is indicated, $F_t(s - |s, S)$ depends only on $S - s$. Define $F_t(x| - \infty, S)$ in the same way so that $F_t(x| - \infty, S) = 1 - \Phi^{(t)}(S - x -)$,* and let $r_t(S - s) = 0 = F_t(- \infty| - \infty, S)$ when $s = - \infty$. The expected cost for the n-period problem is then

$$f_n(s, S) = G(S) + \sum_{t=1}^{n-1} \alpha^t \left[(K + G(S)) r_t(S - s) + \int_{s-}^{S} G(x) dF_t(x|s, S) \right]. \tag{1}$$

For the sake of compactness the convention is adopted here that a summation vanishes if its lower limit exceeds its upper limit. Since $G(x)$ is continuous and nonnegative the integrals in (1) exist and $f_n(s, S)$ is defined. When s is finite $f_n(s, S)$ is finite for all n. For the policy $(- \infty, S)$, however, some of the integrals may be infinite and then $f_n(- \infty, S)$ will be also. Note that if an initial set-up cost is included $f_n(s, S)$ will be increased by only a constant and so the minimization problem is unchanged. Using Greenberg's recursive relationships for $F_t(s|s, S)$ and $r_t(S - s)$ one obtains:

$$\begin{aligned} F_t(x|s, S) &= 1 - \Phi^{(t)}(S - x -) - \sum_{i=1}^{t-1} r_i(S - s) \Phi^{(t-i)}(S - x -) \\ &\quad - \int_{x}^{s-} \Phi(v - x) d\Phi^{(t-1)}(S - v) - \sum_{i=1}^{t-2} r_i(S - s) \int_{x}^{s-} \Phi(v - x) d\Phi^{(t-i-1)}(S - v), \end{aligned} \tag{2}$$

$$r_t(S - s) = 1 - \Phi^{(t)}(S - s) - \sum_{i=1}^{t-1} r_i(S - s) \Phi^{(t-i)}(S - s),$$

for $t \geq 1$ where $\Phi^{(t)}(S - s) = 1$ when $s = - \infty$.

Now use the assumption that $\Phi(\cdot)$ possesses a density $\phi(\cdot)$ and let $\phi^{(t)}(\cdot)$ denote its t-fold convolution with itself. Under mild conditions it has been shown [1, chap. 14 and 15] that for finite s,

*$\Phi^{(t)}(\cdot)$ is the t-fold convolution of $\Phi(\cdot)$ with itself.

$F_t(x|s, S)$ converges in the sense of distributions to a stationary distribution $F(x|s, S)$ satisfying in Greenberg's notation

$$F(x|s, S) = 1 - (1 + M(S-s))^{-1}(M(S-x) - \int_x^s m(S-v)\Phi(v-x)dv),$$

where

$$m(x) = \sum_{t=1}^{\infty} \varphi^{(t)}(x)$$

and

$$M(x) = \sum_{t=1}^{\infty} \Phi^{(t)}(x)$$

are the renewal density and renewal functions, respectively. Thus, as $t \to \infty$

$$r_t(S-s) \to r(S-s) \equiv F(s|s, S)$$

$$= (1+M(S-s))^{-1}.$$

As $n \to \infty$ with $\alpha < 1$, $f_n(s, S)$ converges to

$$f(s, S) \equiv G(S) + \sum_{t=1}^{\infty} \alpha^t \left[(K+G(S))r_t(S-s) + \int_s^S G(x)dF_t(x|s, S) \right],$$

which may be ∞ if $s = -\infty$. If $\alpha = 1$ then it is shown in section 4 that because of the convergence in distribution for finite s, $k_n(s, S) \equiv \frac{1}{n} f_n(s, S)$ converges to

$$k(s, S) \equiv (K+G(S))r(S-s) + \int_s^S G(x)dF(x|s, S).$$

Furthermore, that $k_n(-\infty, S)$ is unbounded as $n \to \infty$ is seen as follows. Let L be an arbitrarily large number and choose $A < S$ so that $\min_{y \leq A} G(y) \geq L$. Since demand is nonnegative there exists an $N = N(L)$ such that for all $t > N$, $\Phi^{(t)}(S-A) < L^{-1}$.
Hence, since $G(y) > 0$

$$\liminf k_n(-\infty, S) = \liminf \frac{1}{n} \sum_{t=N+1}^{n} \int_{-\infty}^{S} G(x)\varphi^{(t)}(S-x)dx$$

$$\geq \liminf \frac{1}{n} \sum_{t=N+1}^{n} \int_{-\infty}^{A} G(x)\varphi^{(t)}(S-x)dx$$

$$\geq \lim \frac{n-N}{n} \left(1 - \frac{1}{L}\right) L$$

$$= L - 1.$$

Because L is arbitrary it is concluded that $k_n(-\infty, S) \to k(-\infty, S) \equiv \infty$.

While Greenberg [6] refers to analysis and computations for this model we aren't aware of their availability. Deb and Sargent [4] perform some general analysis and then present calculations for this model with exponential demand, linear inventory costs, $\alpha = 1$, $x_1 = 0$, and s chosen nonnegative. Their figures indicate that $f_n(s, S)$, $f(s, S)$, and $k(s, S)$ are very flat near the optimal values. Our following analysis concerns properties of the solutions to the optimization problem encountered in this general model.

The following lemma establishes continuity of costs as functions of policies.

LEMMA 1: The functions $f_n(s, S)$, $f(s, S)$, and $k(s, S)$ are continuous in (s, S). For fixed S as $s \to -\infty$, $f_n(s, S) \to f_n(-\infty, S)$ and $f(s, S) \to f(-\infty, S)$. Furthermore, as $s \to -\infty$, $k(s, S)$ is uniformly unbounded for S in any finite interval.

PROOF: For all t, $F_t(x \mid s, S)$ is seen to be absolutely continuous in x and continuous in (s, S) from the recursive expression (2). Thus, each term in (1) is continuous in (s, S) and so $f_n(s, S)$ is. If R is that part of an arbitrary bounded rectangle having $s \leq S$, then for all n

$$\sup_{(s,S)\in R} |f_{n+1}(s, S) - f_n(s, S)| = \sup_{(s,S)\in R} \alpha^n[(K + G(S))r_n(S - s) + \int_s^S G(x)dF_n(x \mid s, S)] \leq \alpha^n(K + \sup_{a \leq x \leq b} G(x)),$$

where a and b are the smallest s and largest S, respectively, in R. Since $G(x)$ is continuous its supremum over $[a, b]$ is finite. Thus, it is concluded that $f_n(s, S)$ converges uniformly to $f(s, S)$ in R, and so the latter must also be continuous. Finally, since $F(x|s, S)$ is absolutely continuous in x and continuous in (s, S), $k(s, S)$ is continuous.

Next, for fixed S as $s \to -\infty$, $r_i(S - s) \to 0$ for $i \geq 1$. It will first be shown that $f(s, S) \to f(-\infty, S)$ as $s \to -\infty$. To this end suppose that $f(-\infty, S) < \infty$, recalling that

$$f(-\infty, S) = G(S) + \sum_{t=1}^{\infty} \alpha^t \int_{-\infty}^{S} G(x)dF_t(x \mid -\infty, S).$$

By (2) for $x \geq s$

$$F_t(x|s, S) = F_t(x| - \infty, S) - \sum_{i=1}^{t-1} r_i(S - s)(1 - F_{t-i}(x| - \infty, S)), \tag{3}$$

so that

$$\begin{aligned} f(s, S) &= G(S) + \sum_{t=1}^{\infty} \alpha^t \left[(K + G(S))r_t(S - s) + \int_s^S G(x)dF_t(x| - \infty, S) \right. \\ &\qquad \left. + \sum_{i=1}^{t-1} r_i(S - s) \int_s^S G(x)dF_{t-i}(x| - \infty, S) \right] \\ &\leq f(-\infty, S) + \sum_{i=1}^{\infty} \alpha^i r_i(S - s) \left[K + G(S) + \sum_{i+1}^{\infty} \alpha^{t-i} \int_{-\infty}^{S} G(x)dF_{t-i}(x| - \infty, S) \right] \\ &\leq f(-\infty, S) + (1 - \alpha)^{-1}\alpha(K + f(-\infty, S)). \end{aligned}$$

Thus, by the dominated convergence theorem

$$\lim_{s \to -\infty} f(s, S) = f(-\infty, S).$$

If $f(-\infty, S) = \infty$ notice first that by (3) since each $F_{t-i}(x| -\infty, S)$ is monotone nondecreasing in x

$$\int_s^S G(x)dF_t(x|s, S) \geqslant \int_s^S G(x)dF_t(x| -\infty, S).$$

Hence, for all $s \leqslant S$

$$f(s, S) \geqslant G(S) + \sum_{t=1}^{\infty} \alpha^t \int_s^S G(x)dF_t(x| -\infty, S),$$

and by Fatou's lemma $f(s, S) \to \infty$ as $s \to -\infty$. The same proof applies to $f_n(s, S)$ for $n \geqslant 1$ if $n-1$ replaces ∞ as the upper limit of the summations.

It remains to show that for all finite A and B and large L there exists an s_0 such that for all $s \leqslant s_0, k(s, S) \geqslant L$ for all $S \in [A, B]$. First choose $C < A$ so that

$$\min_{y \leqslant C} G(y) \geqslant (L-1)^{-1}L^2,$$

then choose $s_0 \leqslant C$ so that $M(A-s_0) > 0$

$$[(M(A-s_0))^{-1}+1]^{-1} \geqslant 1-(2L)^{-1},$$

and

$$M(B-C)(1+M(A-s_0))^{-1} \leqslant (2L)^{-1}.$$

For any $s \leqslant s_0$ since $G(x)$ is positive

$$k(s, S) = (1+M(S-s))^{-1}\left(K+G(S)+\int_s^S G(x)m(S-x)dx\right)$$

$$\geqslant (1+M(S-s))^{-1}\int_s^C G(x)m(S-x)dx$$

$$\geqslant (1+M(S-s))^{-1}(M(S-s)-M(S-C))(L-1)^{-1}L^2$$

$$\geqslant ([(M(A-s_0))^{-1}+1]^{-1}-(1+M(A-s_0))^{-1}M(B-C))(L-1)^{-1}L^2$$

$$\geqslant (1-L^{-1})(L-1)^{-1}L^2$$

$$= L. \qquad \text{Q.E.D.}$$

3. POLICY BOUNDS

It is useful to establish bounds within which there exist optimal values of (s, S). Since $G(x)$ need not be quasi-convex the bounds of Veinott and Wagner [14] may not hold. Define $\underline{S}$ and $\underline{s}$ as they do:

$$\underline{S} = \min\{x \mid G(x) = \min_{y} G(y)\},$$

$$\underline{s} = \min\{x \mid G(x) = K + G(\underline{S})\};$$

but let $\bar{s}$ satisfy the following:

$$\bar{s} = \max\{x \mid G(x) = K + G(\underline{S})\}.$$

Also let $\eta \in (0,1)$ be an otherwise arbitrary fixed constant. Recalling that $K + G(\underline{S}) > 0$ define $\underline{\underline{s}}$ and $\bar{\bar{s}}$ as follows:

$$\underline{\underline{s}} = \min\{x \mid G(x) = (K + G(\underline{S}))/\eta\}$$

$$\bar{\bar{s}} = \max\{x \mid G(x) = (K + G(\underline{S}))/\eta\}.$$

Also, let (s_n, S_n) indicate an optimal value of (s, S) for the n-period problem, $f_n \equiv f_n(s_n, S_n)$, and $k_n = f_n/n$. Similarly s^* and S^* will be optimal for the infinite-horizon problem, $f = f(s^*, S^*)$, and $k = k(s^*, S^*)$. Attention will be restricted to demonstrating bounds for some optimal values of the parameters, although in many situations every optimal value will satisfy them because they are weak. Recall that here the parameters (s_n, S_n) are used in every one of the n periods, whereas in an arbitrary (s, S) policy a sequence of parameters ——— $(s_1, S_1), (s_2, S_2), \ldots, (s_n, S_n)$ ——— is used.

LEMMA 2: For all n and every (s_n, S_n), $s_n \leq \bar{s}$ and $S_n \geq \underline{s}$. Also, $s^* \leq \bar{s}$ and $S^* \geq \underline{s}$ for every (s^*, S^*).

PROOF: If $s > \bar{s}$ or $S < \underline{s}$

$$k(s, S) = (K + G(S))r(S - s) + \int_{s}^{S} G(x)\,dF(x \mid s, S)$$

$$> K + G(\underline{S})$$

$$= k(\underline{S}, \underline{S})$$

$$\geq k.$$

The proof for a finite horizon or for $\alpha < 1$ is clear since each term in the summation in $f_n(s, S)$ or $f(s, S)$ is of the above form. Q.E.D.

For $\alpha < 1$ let $\bar{S}(\alpha)$ be the largest S such that

$$G(S) = f(\underline{S}, \underline{S}).$$

Still denoting the mean of ξ by μ let n_0 be an integer satisfying

$$(\bar{\bar{s}}-\underline{s})\mu^{-1}+\eta<(1-\eta)n_0$$

and choose S_0 so that

$$\Phi^{(n_0)}(S_0-\bar{\bar{s}})>\eta.$$

By Blackwell's theorem [5] there is an $\bar{S}_1$ so that for all $S\geq\bar{S}_1$

$$M(S-\underline{s})-M(S-\bar{\bar{s}})<(\bar{\bar{s}}-\underline{s})\mu^{-1}+\eta.$$

Furthermore, for an appropriate $\bar{S}_2$ and any $S\geq\bar{S}_2$

$$(M(S-\underline{s})-M(S-\bar{\bar{s}}))(1+M(S-\bar{\bar{s}}))^{-1}<1-\eta,$$

because the numerator is bounded. Now define $\bar{S}(1)=\max(S_0,\bar{S}_1,\bar{S}_2)$.

LEMMA 3: For all α and n every $S_n\leq\bar{S}(\alpha)$ and every $S^*\leq\bar{S}(\alpha)$.

PROOF: When $\alpha<1$ for any $S>\bar{S}(\alpha)$ and any $s\geq-\infty$

$$\begin{aligned}f(s,S)&>f_n(s,S)\\&>G(S)\\&>f(\underline{S},S)\\&>f_n(\underline{S},S).\end{aligned}$$

Hence, $S_n\leq\bar{S}(\alpha)$ and $S^*\leq\bar{S}(\alpha)$.

When $\alpha=1$ it is convenient to express $f_n(s,S)$ in terms of the expected cost incurred in each of the inventory cycles begun with an order returning the level to S. Let $\{y_t\}$ be the random sequence of levels generated by the policy (s,S) so that

$$f_n(s,S)=\sum_{t=1}^{n}P[y_t=S]\left\{K+G(S)+\sum_{i=1}^{n-t}\int_s^S G(x)\varphi^{(i)}(S-x)\,dx\right\}-K. \tag{4}$$

Suppose that $S>\bar{S}(1)$ and, by Lemma 2, $s\leq\bar{s}$. If $1\leq n-t\leq n_0$ the term in braces in (4) is larger than

$$K+G(\underline{S})+\sum_{i=1}^{n-t}(K+G(\underline{S}))\eta^{-1}\Phi^{(i)}(S-\bar{\bar{s}})>(n-t+1)(K+G(\underline{S})),$$

since $\Phi^{(i)}(S-\bar{\bar{s}})\geq\Phi^{(n_0)}(S-\bar{\bar{s}})$ for $i\leq n_0$. On the other hand, if $n-t>n_0$ a lower bound for this term is

$$K+G(\underline{S})+(K+G(\underline{S}))\sum_{i=1}^{n-t}(\Phi^{(i)}(S-s)-\Phi^{(i)}(S-\max(s,\underline{s}))+(K+G(\underline{S}))\eta^{-1}\sum_{i=1}^{n-t}\Phi^{(i)}(S-\bar{\bar{s}})$$

$$\geqslant (K+G(\underline{S}))\left[1+\sum_{i=1}^{n-t}(\Phi^{(i)}(S-s)-\Phi^{(i)}(S-\max(s,\underline{s}))+\Phi^{(i)}(S-\bar{\bar{s}}))\right]$$

$$+(K+G(\underline{S}))\eta^{-1}(1-\eta)\sum_{i=1}^{n_0}\Phi^{(i)}(S-\bar{\bar{s}}).$$

The last term in this expression is larger than

$$(K+G(\underline{S}))(1-\eta)n_0 > (K+G(\underline{S}))(M(S-\underline{s})-M(S-\bar{\bar{s}}))$$

$$\geqslant (K+G(\underline{S}))\sum_{i=1}^{n-t}(\Phi^{(i)}(S-\underline{s})-\Phi^{(i)}(S-\bar{\bar{s}}))$$

$$\geqslant (K+G(\underline{S}))\sum_{i=1}^{n-t}(\Phi^{(i)}(S-\max(s,\underline{s}))-\Phi^{(i)}(S-\bar{\bar{s}})).$$

Hence, the term in braces in (4) is strictly greater than

$$(K+G(\underline{S}))[1+\sum_{i=1}^{n-t}\Phi^{(i)}(S-s)],$$

which is the corresponding contribution to $f_n(\underline{S}, \underline{S})$ independent of y_t, so that $S_n \leqslant \bar{S}(1)$.

To consider $k(s, S)$ for $S > \bar{S}(1)$ and $s \leqslant \bar{s}$ note that

$$F(\bar{\bar{s}}|s,S)-F(\max(s,\underline{s})|s,S)=[M(S-\max(s,\underline{s}))-M(S-\bar{\bar{s}})][1+M(S-s)]^{-1}$$

$$\leqslant [M(S-\underline{s})-M(S-\bar{\bar{s}})][1+M(S-s)]^{-1}$$

$$< 1-\eta.$$

Thus,

$$k(s,S)=(K+G(S))r(S-s)+\int_s^S G(x)dF(x|s,S)$$

$$> (K-G(\underline{S}))\eta^{-1}[1-F(\bar{\bar{s}}|s,S)+F(\max(s,\underline{s})|s,S)]$$

$$> K+G(\underline{S})$$

$$> k(\underline{S},\underline{S}).$$

Hence, $S^* \leqslant \bar{S}(1)$. Q.E.D.

Whenever α is considered fixed in what follows $\bar{S}(\alpha)$ may be denoted simply as $\bar{S}$.

Finding a lower limit for optimal values of s_n is a more difficult problem. One reason is that if ξ is a bounded random variable then when K is large enough it is optimal to take $s_n = -\infty$ for small n. Another is that even when ξ is unbounded $f_n(-\infty, \underline{S}) < f_n(\underline{S}, \underline{S})$ for appropriate n, K, and $\Phi(\cdot)$.

The following lemma, while not giving a lower bound for all s_n, shows that the parameter may be chosen finite.

LEMMA 4: For all $n \geq 2$ there exists an $\underline{s}_n$ such that for all S either $f_n(-\infty, S) > f_n$ or $f_n(-\infty, S) = f_n(s, S)$ for all $s \leq \underline{s}_n$. The latter occurs only if $\Phi^{(n-1)}(S-\underline{s}_n)=1$.

PROOF: For each n consider any S such that $f_n(-\infty, S)=f_n$, so that $\underline{s} \leq S \leq \bar{S}$.

For $n=2$ note that

$$f_2(-\infty, S)=G(S)+\alpha\int_{-\infty}^{S} G(x)\,\varphi(S-x)dx,$$

and

$$f_2(s, S)=G(S)+\alpha\int_{s}^{S} G(x)\varphi(S-x)dx+\alpha\int_{-\infty}^{s}(K+G(S))\,\varphi(S-x)dx,$$

so that

$$f_2(-\infty, S)-f_2(s, S)=\alpha\int_{-\infty}^{s}(G(x)-K-G(S))\,\varphi(S-x)dx.$$

Since $G(x)\to\infty$ as $x\to-\infty$ there exists an $\underline{s}_2$ such that for all $x \leq \underline{s}_2$

$$G(x) > \max_{\underline{s}\leq S\leq \bar{S}}\{K+G(S)\}+\eta;$$

and so when $s \leq \underline{s}_2$

$$f_2(-\infty, S) \geq f_2(s, S).$$

Thus, equality must hold, and this can happen only if $\Phi(S-\underline{s}_2)=1$.

To consider an arbitrary $n \geq 3$ let $C_t(s, S)$ be the expected cost incurred in period t when (s, S) is followed so that for all s

$$f_n(s, S)=\sum_{t=1}^{n}\alpha^{t-1}C_t(s, S).$$

Let period T_t be the last period before period $t+1$ in which an order is placed; if there has been no ordering let $T_t=1$. Thus, for $t \geq 2$ and any $s < S$ so that $\Phi(S-s) > 0$

$$C_t(s, S)=\int_{s}^{S} G(x)\varphi^{(t-1)}(S-x)dx$$

$$+\sum_{j=2}^{t-1}\int_{-\infty}^{s}\int_{s}^{S} G(x)\,(\Phi^{(t-j)}(S-s))^{-1}\varphi^{(t-j)}(S-x)dx\,d_yP\left[\sum_{i=1}^{j-1}\zeta_i \geq S-y,\ T_t=j\right]$$

$$+\int_{-\infty}^{s}(K+G(S))\,d_yP\left[\sum_{i=1}^{t-1}\zeta_i \geq S-y,\ T_t=t\right].$$

In terms of the same T_t

$$C_t(-\infty, S)=\int_{-\infty}^{S} G(x)\,\varphi^{(t-1)}(S-x)dx=\int_{s}^{S} G(x)\varphi^{(t-1)}(S-x)dx$$

$$+\sum_{j=2}^{t-1}\int_{-\infty}^{s}\int_{s}^{S} G(x+y-S)\,(\Phi^{(t-j)}(S-s))^{-1}\varphi^{(t-j)}(S-x)\,dx\,d_yP\left[\sum_{i=1}^{j-1}\zeta_i \geq S-y,\ T_t=j\right]$$

$$+\int_{-\infty}^{s} G(y)\,d_yP\left[\sum_{i=1}^{t-1}\zeta_i \geq S-y,\ T_t=t\right].$$

Hence,

(5) $C_t(-\infty, S)-C_t(s, S)=$

$$\sum_{j=2}^{t-1}\int_{-\infty}^{s}\int_{s}^{S}[G(x+y-S)-G(x)]\,(\Phi^{(t-j)}(S-s))^{-1}\varphi^{(t-j)}(S-x)\,dx\,d_yP\left[\sum_{i=1}^{j-1}\zeta_i \geq S-y,\ T_t=j\right]$$

$$+\int_{-\infty}^{s}[G(y)-K-G(S)]\,d_yP\left[\sum_{i=1}^{t-1}\zeta_i \geq S-y,\ T_t=t\right].$$

For $s \leq \underline{s}_2$ the last term is nonnegative, so taking any s_0 such that $s < s_0 < \underline{s}_2$ and suppressing the unchanging argument of the second probability measure consider an arbitrary term in the summation:

$$\int_{-\infty}^{s}\left[\int_{s}^{s_0}[G(x+y-S)-G(x)]\,(\Phi^{(t-j)}(S-s))^{-1}\varphi^{(t-j)}(S-x)\,dx\right.$$

$$\left.+\int_{s_0}^{S}[G(x+y-S)-G(x)]\,(\Phi^{(t-j)}(S-s))^{-1}\varphi^{(t-j)}(S-x)\,dx\right]d_yP.$$

The first of the inner integrals is larger than

$$\int_{s}^{s_0}[G(\underline{S})-G(x)]\,(\Phi^{(t-j)}(\underline{s}-s))^{-1}\varphi^{(t-j)}(S-x)\,dx \geq -\int_{-\infty}^{s_0}[G(x)-G(\underline{S})](\Phi^{(t-j)}(\underline{s}-s))^{-1}\varphi^{(t-j)}(S-x)dx,$$

which is finite since $f_n(-\infty, S)=f_n$. Thus, there exists an $s_0=s_0(\eta, t-j)$ such that

$$\int_{-\infty}^{s_0}[G(x)-G(\underline{S})](\Phi^{(t-j)}(\underline{s}-s_0))^{-1}\varphi^{(t-j)}(S-x)\,dx < \eta.$$

Now, for all $x \in [s_0, S]$

$$G(x+y-S)-G(x) \geq G(x+y-S)-\max_{s_0 \leq x \leq S} G(x)$$

$$\geq 2\eta(\Phi^{(t-j)}(\underline{s}-s_0))^{-1}$$

for all $y \leq$ some $s(\eta, t-j) \leq s_0(\eta, t-j)$. Hence, for all $s \leq s(\eta, t-j)$ each term in the summation is bounded below by

$$\int_{-\infty}^{s}\eta\,d_yP \geq 0.$$

Thus, for $s \leq \min_{2 \leq j \leq t-1} \{s(\eta, t-j)\}$ when $t \geq 3$

$$C_t(-\infty, S) \geq C_t(s, S).$$

Also, since $s_0 \leq s_2$, $C_2(-\infty, S) \geq C_2(s, S)$. If $\underline{s}_n$ is the smallest of these minima for $n \geq t \geq 3$ then for $s \leq \underline{s}_n$

$$f_n = f_n(-\infty, S) \geq f_n(s, S) \geq f_n.$$

Hence, equality holds throughout these relations. However, by (5) $C_n(-\infty, S) = C_n(s, S)$ only if $\Phi^{(n-1)}(S-s) = 1$. Q.E.D.

Veinott and Wagner's renewal approach to the stationary analysis is useful for studying $f(s, S)$. Their ideas will be employed with slight notational changes. Let $M_\alpha(x)$ and $m_\alpha(x)$ be the discount renewal function and its density:

$$m_\alpha(x) = \sum_{t=1}^{\infty} \alpha^t \varphi^{(t)}(x)$$

$$M_\alpha(x) = \sum_{t=1}^{\infty} \alpha^t \Phi^{(t)}(x).$$

Let T_l now denote the random period in which the inventory level falls below s for the lth time when (s, S) is followed in the infinite-horizon problem. The dependence of T_l on the policy need not be expressed here since specific parameters s and S will be chosen. Since $\Phi(0) = 0$ it follows that T_l is a proper random variable for all $l \geq 1$ if $s > -\infty$. Also, let $T_0 = 0$. The total expected cost then may be written as the sum of the expected discounted costs during each cycle begun with an order and continuing until the next order is placed; i.e.,

$$f(s, S) = \sum_{l=1}^{\infty} E[\alpha^{T_l - 1} C_l] - K$$

where

$$C_l = K + G(S) + \sum_{t=1}^{\infty} \alpha^t \int_0^{S-s} G(S-x) \varphi^{(t)}(x)\, dx$$

$$= K + G(S) + \int_0^{S-s} G(S-x) m_\alpha(x)\, dx. \tag{6}$$

Now C_l is a constant independent of l and for $l \geq 1$

$$E[\alpha^{T_l}] = E[E[\alpha^{T_l} | T_{l-1}]]$$

$$= E[\alpha^{T_{l-1}} E[\alpha^{T_l - T_{l-1}} | T_{l-1}]]$$

$$= E[\alpha^{T_{l-1}} E[\alpha^{T_1}]]$$

$$= E[\alpha^{T_1}] E[\alpha^{T_{l-1}}].$$

Thus, by induction for $l \geq 0$

$$E[\alpha^{T_l}] = (E[\alpha^{T_1}])^l.$$

As Veinott and Wagner point out, letting $\Phi^{(0)}(x) = 1$ for $x \geq 0$

$$E[\alpha^{T_1}] = \sum_{l=1}^{\infty} \alpha^l [\Phi^{(l-1)}(S-s) - \Phi^{(l)}(S-s)]$$

$$= \alpha - (1-\alpha) M_\alpha(S-s).$$

Therefore,

$$f(s, S) = C_1 \sum_{l=1}^{\infty} (E[\alpha^{T_1}])^{l-1} - K$$

$$= C_1[(1-\alpha)(1+M_\alpha(S-s))]^{-1} - K. \tag{7}$$

LEMMA 5: For all α every s^* is finite. Furthermore, for $\alpha = 1$ there exists an $\underline{s}^*$ such that all $s^* \geq \underline{s}^*$.

PROOF: For $\alpha = 1$ any $S^* \in [\underline{s}, \bar{S}]$ by Lemmas 2 and 3. The existence of $\underline{s}^*$ is then ensured by Lemma 1. Thus, take $\alpha < 1$ and consider any S such that $f(-\infty, S) = f$. Choose $s < \underline{s}$ so that

$$M_\alpha(\underline{s} - s) > \alpha(1-\alpha)^{-1}\eta,$$

$$\min_{y \leq \underline{s}} G(y) > (1 + \alpha(1-\alpha)^{-1}\eta)^{-1}(f+K).$$

In terms of the T_l for the policy (s, S), $f(-\infty, S)$ may be expressed as

$$f(-\infty, S) = G(S) + \sum_{l=1}^{\infty} \alpha^l \int_0^{\infty} G(S-x)\phi^{(l)}(x)dx$$

$$= G(S) + \int_0^{\infty} G(S-x) m_\alpha(x) dx \tag{8}$$

$$= \sum_{l=1}^{\infty} E[\alpha^{T_{l-1}} C_l'],$$

where

$$C_l' = E[G(S - \sum_{i=1}^{T_{l-1}} \xi_i) + \sum_{l=1}^{\infty} \alpha^l \int_0^{S-s} G(S - \sum_{i=1}^{T_{l-1}} \xi_i - x)\varphi^{(l)}(x)dx | T_{l-1}]$$

$$= E[G(S - \sum_{i=1}^{T_{l-1}} \xi_i) + \int_0^{S-s} G(S - \sum_{i=1}^{T_{l-1}} \xi_i - x) m_\alpha(x)dx | T_{l-1}].$$

Now C_1' is really a constant equal to $C_1 - K$ so that

$$f(-\infty, S) - f(s, S) = \sum_{l=2}^{\infty} E[\alpha^{T_l - 1}(C_l' - C_1)].$$

For $l \geq 2$ with probability one

$$C_l' \geq \min_{y \leq s} G(y)[1 + M_a(S-s)]$$

$$\geq \min_{y \leq s} G(y)[1 + M_a(\underline{s}-s)]$$

$$> f + K$$

$$> C_1$$

by (6) and (8). Since all T_l are finite it follows that

$$f(-\infty, S) - f(s, S) > 0,$$

which is a contradiction. Hence, $f(-\infty, S) > f$ and s^* is finite. A similar proof would hold for Lemma 4 although the finiteness of the horizon necessitates truncated versions of the functions and random variables used here. Q.E.D.

Unfortunately it is not evident that any sequence $\{s_n\}$ is bounded below. By Lemma 4 when demand is bounded, in fact, it may be possible to take $s_n = -\infty$ for some n. Also, finiteness of s^* need not imply finiteness of inf $\{s^*\}$ if the number of different values of S^* is infinite.

LEMMA 6: If there exists an s_0 for which

$$\min_{y \leq s_0} G(y) = G(s_0) = \max_{s_0 \leq y \leq \bar{S}} G(y) > K + \max_{\underline{s} \leq y \leq \bar{S}} G(y)$$

then there exists an $\underline{s}^{*\prime}$ such that every $s^* > \underline{s}^{*\prime}$. If also $\Phi^{(n-1)}(\bar{S}(\alpha) - s_0) < 1$ then there exists an $\underline{s}_n'(\alpha)$ such that every $s_n > \underline{s}_n'(\alpha)$.

PROOF: It will first be shown that $\sum_{t=1}^{n-1} \alpha^t r_t(S-s) \leq \sum_{t=1}^{n-1} \alpha^t r_t(S-s')$ for all $n \geq 2$ if $s \leq s' \leq S$. For an arbitrary $n \geq 2$ let N be the number of reorders when (s, S) is followed in the n-period problem. Let $T_0, T_1, \ldots, T_N$ be the periods prior to the ordering, where $T_0 = 0$. Define N' and $T_0', T_1', \ldots, T_N'$, similarly for (s', S). It is evident that with probability one $N \leq N'$ and $T_l' \leq T_l$ for $0 \leq l \leq N$. Therefore,

$$\sum_{t=1}^{n-1} \alpha^t r_t(S-s) = \sum_{t=1}^{n-1} \alpha^t P[\text{some } T_l = t]$$

$$= \sum_{t=1}^{n-1} \alpha^t E\left[\sum_{l=1}^{N} P[T_l = t \mid N, N']\right]$$

$$= E\left[\sum_{l=1}^{N}\sum_{t=1}^{n-1}\alpha^t P\,[T_l = t \mid N, N']\right]$$

$$= E\left[\sum_{l=1}^{N} E[\alpha^{T_l} \mid N, N']\right]$$

$$\leqslant E\left[\sum_{l=1}^{N'} E[\alpha^{T_l} \mid N, N']\right]$$

$$= \sum_{t=1}^{n-1}\alpha^t r_t(S-s').$$

Next, by (2) for $n \geqslant 2$ and $x \geqslant s$

$$\sum_{t=1}^{n-1}\alpha^t F_t(x|s,S) = \sum_{t=1}^{n-1}\alpha^t - \sum_{t=1}^{n-1}\alpha^t\left[\Phi^{(t)}(S-x) + \sum_{i=1}^{t-1} r_i(S-s)\Phi^{(t-i)}(S-x)\right]$$

$$= \frac{\alpha-\alpha^n}{1-\alpha} - \sum_{t=1}^{n-1}\alpha^t\Phi^{(t)}(S-x) - \sum_{i=1}^{n-2}\alpha^i r_i(S-s)\sum_{t=i+1}^{n-1}\alpha^{t-i}\Phi^{(t-i)}(S-x)$$

$$= \frac{\alpha-\alpha^n}{1-\alpha} - \sum_{t=1}^{n-1}\alpha^t\Phi^{(t)}(S-x) - \sum_{i=1}^{n-2}\alpha^i r_i(S-s)\sum_{t=1}^{n-1-i}\alpha^t\Phi^{(t)}(S-x)$$

$$= \frac{\alpha-\alpha^n}{1-\alpha} - \sum_{t=1}^{n-1}\alpha^t\Phi^{(t)}(S-x)\left[1 + \sum_{i=1}^{n-1-t}\alpha^i r_i(S-s)\right]$$

Now, to prove the lemma note that if $\Phi^{(n-1)}(\bar{S}(\alpha)-s_0) < 1$ then there exists an $\underline{s}'_n(\alpha) < s_0$ such that $\Phi^{(n-1)}(\underline{s}-\underline{s}'_n(\alpha)) > \Phi^{(n-1)}(\bar{S}(\alpha)-s_0)$. Let $S \in [\underline{s}, \bar{S}(\alpha)]$ and $s \leqslant \underline{s}'_n(\alpha)$. By (1)

$$f_n(s,S) - f_n(s_0,S) = \sum_{t=1}^{n-1}\alpha^t\Big[(K+G(S))(r_t(S-s) - r_t(S-s_0))$$

$$+ \int_{s_0}^{S} G(x)\,d(F_t(x|s,S) - F_t(x|s_0,S)) + \int_{s}^{s_0} G(x)\,dF_t(x|s,S)\Big]$$

$$= (K+G(S))\sum_{t=1}^{n-1}\alpha^t(r(S-s) - r_t(S-s_0)) + \int_{s_0}^{S} G(x)\,d\Big[\sum_{t=1}^{n-1}\alpha^t\,(F_t(x|s,S)$$

$$- F_t(x|s_0,S))\Big] + \sum_{t=1}^{n-1}\alpha^t\int_{s}^{s_0} G(x)\,dF_t(x|s,S)$$

$$= (K+G(S))\left(\sum_{t=1}^{n-1}\alpha^t r_t(S-s) - \sum_{t=1}^{n-1}\alpha^t r_t(S-s_0)\right)$$

$$+ \sum_{t=1}^{n-1}\alpha^t\int_{s_0}^{S} G(x)\varphi^{(t)}(S-x)\left(\sum_{i=1}^{n-1-t}\alpha^i r_i(S-s) - \sum_{i=1}^{n-1-t}\alpha^i r_i(S-s_0)\right)dx$$

$$+\sum_{t=1}^{n-1}\alpha^t\int_s^{s_0}G(x)dF_t(x|s,S)$$

$$\geqslant G(s_0)\left[\sum_{t=1}^{n-1}\alpha^t(r_t(S-s)-r_t(S-s_0))+\int_{s_0}^{S}\sum_{t=1}^{n-1}\alpha^t\varphi^{(t)}(S-x)\sum_{i=1}^{n-1-t}\alpha^i(r_i(S-s)\right. \tag{9}$$

$$\left.-r_i(S-s_0))dx+\sum_{t=1}^{n-1}\alpha^t\int_s^{s_0}dF_t(x|s,S)\right]$$

$$=G(s_0)\sum_{t=1}^{n-1}\alpha^t[r_t(S-s)-r_t(S-s_0)+F_t(S|s,S)-F_t(S|s_0,S)$$

$$-F_t(s_0|s,S)+F_t(s_0|s_0,S)+F_t(s_0|s,S)-F_t(s|s,S)]$$

$$=0.$$

Finally, the condition on $\underline{s}'_n(\alpha)$ ensures that $\Phi^{(n-1)}(S-s)>\Phi^{(n-1)}(S-s_0)$. By inspection of the proof of the first inequality here $\sum_{t=1}^{n-1}r_t(S-s_0)>\sum_{t=1}^{n-1}r_t(S-s)$, so that the inequality in (9) is strict. Thus, $f_n(s,S)>f_n(s_0,S)\geqslant f_n$ and $s_n>\underline{s}'_n(\alpha)$.

For the infinite-horizon problem the above argument applies with an appeal to the dominated convergence theorem. A more direct proof, however, employs the discount renewal function. First, there exists an $\underline{s}^{*\prime}$ such that $M_\alpha(\underline{s}-\underline{s}^{*\prime})>M_\alpha(\bar{S}-s_0)$ for all α since $\Phi(0)=0$. Let $S\in[\underline{s},\bar{S}]$ and $s\leqslant\underline{s}^{*\prime}$, so that $M_\alpha(S-s)>M_\alpha(S-s_0)$. By (7) then

$$(1-\alpha)(f(s,S)-f(s_0,S))=\left(K+G(S)+\int_0^{S-s}G(S-x)m_\alpha(x)dx\right)(1+M_\alpha(S-s))^{-1}$$

$$-\left(K+G(S)+\int_0^{S-s_0}G(S-x)m_\alpha(x)dx\right)(1+M_\alpha(S-s_0))^{-1}$$

$$=\left(K+G(S)+\int_0^{S-s_0}G(S-x)m_\alpha(x)dx\right)[(1+M_\alpha(S-s))^{-1}$$

$$-(1+M_\alpha(S-s_0))^{-1}]+(1+M_\alpha(S-s))^{-1}\int_{S-s_0}^{S-s}G(S-x)m_\alpha(x)dx$$

$$>G(s_0)(1+M_\alpha(S-s_0))[(1+M_\alpha(S-s))^{-1}-(1+M_\alpha(S-s_0))^{-1}]$$

$$+G(s_0)(1+M_\alpha(S-s))^{-1}(M_\alpha(S-s)-M_\alpha(S-s_0))$$

$$=G(s_0)(1+M_\alpha(S-s))^{-1}[1+M_\alpha(S-s_0)-1-M_\alpha(S-s)$$

$$+M_\alpha(S-s)-M_\alpha(S-s_0)]$$

$$=0.$$

Thus, $f(s, S) > f(s_0, S) \geq f_n$ so that $s^* > \underline{s}^{*\prime}$. Q.E.D.

If an s_0 satisfying the lemma exists then for large enough n an $\underline{s}_n$ will also exist since $\Phi(0) = 0$. Hence, in this case every sequence $\{s_n\}$ will eventually be bounded below. Also, the inequality on $G(s_0)$ can be weakened to require only that

$$G(s_0) > K + \max\{G(y) | y = S_1, S_2, \ldots, S^*\},$$

but this would be of limited use since the S_n and S^* are unknown at the beginning of the problem. Finally, it is important to note that an s_0 can be found if $G(x)$ is quasi-K-convex.

4. VALUE AND POLICY CONVERGENCE

For $\alpha < 1$ it was shown in the proof of Lemma 1 that $f_n(s, S) \to f(s, S)$ uniformly for (s, S) in any R consisting of that part of a bounded rectangle in which $s \leq S$. For $\alpha = 1$ it has been indicated only that $k_n(s, S) \to k(s, S)$ pointwise. The previously omitted proof is as follows: the sequence of summands converges to the indicated limit by the convergence in distribution and the Helly-Bray Lemma; then, this convergence implies convergence in Cesàro mean, i.e., $k_n(s, S) \to k(s, S)$. If the convergence of summands were uniform in (s, S) then the convergence of $k_n(s, S)$ would be also. The standard use of the Helly-Bray Lemma in obtaining uniform convergence of expectations, however, requires a fixed sequence of distributions and an equicontinuous family of functions. In the present situation the sequence is parameterized (by (s, S)), and the family of functions contains either $G(x)$ alone or versions of $G(x)$ which vanish outside the interval (s, S) and therefore are not even continuous. Feller's proof of his version of the Helly-Bray Lemma [5, page 244], however, will be modified to show uniform convergence of the integrals in (1) when the convergence in distribution is uniform in the parameters over any R as above.

LEMMA 7: If $F_t(x|s, S) \to F(x|s, S)$ uniformly in (s, S) over any R and in $x \epsilon [s, S]$ then $k_n(s, S) \to k(s, S)$ uniformly over any R.

PROOF: By the hypothesis and continuity of $G(y)$ it is seen that $(K + G(S))r_t(S - s) \to (K + G(S))r(S - s)$ uniformly in any R. For any function $A(x)$ let $A(x|s, S) = A(x)$ for $s \leq x \leq S$ and vanish elsewhere. Since $G(x)$ is uniformly continuous on the interval bounded by the smallest s and the largest S in R, this interval may be partitioned into a finite number of subintervals ——$[a_0, a_1]$, $[a_1, a_2]$, $\ldots$, $[a_{r-1}, a_r]$——within each of which $G(x)$ oscillates by less than ϵ. Let $H_\epsilon(x)$ be defined on each subinterval $[a_{k-1}, a_k]$ as the smallest value m_k of $G(x)$ on that subinterval. Let

$$E_t[A(x)|s, S] = \int_{-\infty}^{\infty} A(x|s, S)\, dF_t(x|s, S)$$

$$= \int_s^S A(x)\, dF_t(x|s, S),$$

if it has meaning, and let $E[A(x)|s, S]$ be similarly defined. In the standard approach then

$$\left| \int_s^S G(x)\, dF_t(x|s, S) - \int_s^S G(x)\, dF(x|s, S) \right| \leq |E_t[G(x) - H_\epsilon(x)|s, S]|$$

$$+ |E_t[H_\epsilon(x)|s, S] - E[H_\epsilon(x)|s, S]| + |E[G(x) - H_\epsilon(x)|s, S]|.$$

The first and third expressions with absolute value signs on the right are each less than ϵ independent of (s, S). The second expression is bounded above by

$$\sum m_k[|F_t(a_k|s, S)-F(a_k|s, S)|+|F_t(a_{k-1}|s, S)-F(a_{k-1}|s, S)|]$$

where the summation is over those k such that $s \leq a_k \leq S$. Since this is less than ϵ for large enough t independent of (s, S) in R it is concluded that

$$(K+G(S))r_t(S-s)+\int_s^S G(x)dF_t(x|s, S) \to (K+G(S))r(S-s)+\int_s^S G(x)dF(x|s, S)$$

uniformly. Thus, $k_n(s, S) \to k(s, S)$ uniformly in R. Q.E.D.

THEOREM 1: If some $\{s_n\}$ is bounded below then $f_n \to f$. If also $k_n(s, S) \to k(s, S)$ uniformly over any R then $k_n \to k$.

PROOF: Let R be large enough to include $\bar{S}$, $\underline{s}$, $\bar{s}$, and the $\{s_n\}$ in the hypothesis. Then there exists an $N(\eta)$ such that for all $n \geq N(\eta)$ and all S_n associated with s_n

$$f(s_n, S_n) \leq f_n+\eta.$$

Since η is arbitrary it is concluded that

$$\liminf f(s_n, S_n) \leq \liminf f_n.$$

On the other hand,

$$\limsup f_n \leq \limsup f_n(s^*, S^*)$$

$$=f$$

$$\leq f(s_n, S_n)$$

for all n. Therefore, $\limsup f_n \leq \liminf f_n$, so that equality holds throughout and $f_n \to f$. If $k_n(s, S)$ converges uniformly over R the same proof applies. Q.E.D.

The condition on $\{s_n\}$ may be weakened to requiring only that some $\{s_n\}$ have finite limit points. Since all $\{S_n\}$ are bounded there are then limit points of $\{(s_n, S_n)\}$ for all $\{S_n\}$ and the same proof applies. Furthermore, by continuity of $f(s, S)$ and $k(s, S)$ these limit points are optimal for the infinite-horizon problem. These results are analogous to those of Iglehart for $\alpha<1$ [9]. When $G(x)$ is quasi-convex and $\alpha<1$, limit points of the optimal nonstationary parameters for a general finite-horizon problem yield optimal stationary parameters for the general infinite-horizon problem. Furthermore, although for the general problem the optimal cost $f(x_1)$ depends on x_1 at least one optimal stationary policy does not, and the set of optimal (s', S') is the set of pairs minimizing $K+f(s, S)$ [14]. Thus, not only will any (s^*, S^*) here be optimal for all $x_1 \leq \min\{s', s^*\}$, as concluded from our earlier remarks, but, in fact, it will be optimal for all x_1. Finally, $f_n(s_n, S_n) \geq f_n(S_n)$, but these costs will be close for large n since they converge to $f=f(S^*)$. In this sense the optimal policies (s_n, S_n) of this paper approximate the optimal (s, S) policies of the general problem.

When $G(x)$ is quasi-convex and $\alpha=1$ the general optimal costs may converge [8, 13], but convergence of policies has not been established. Here, uniform convergence of $k_n(s, S)$ is assumed to prove convergence of optimal costs, but then the optimality of limiting policies is also obtained. The same assumption would yield convergence of policies in the general problem since optimal parameters are bounded [13]. In a discrete problem eventual boundedness of $\{(s_n, S_n)\}$ permits consideration of only a finite number of policies (s, S) for large n. Thus, there exists an $N(\eta)$ such that for all (s_n, S_n), $k(s_n, S_n) \leq k+\eta$ for all $n \geq N(\eta)$ without the explicit assumption of uniform convergence of $k_n(s, S)$. The remainder of the proof of the theorem applies and $k_n \to k$.

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AN INFORMAL SURVEY OF MULTI-ECHELON INVENTORY THEORY*

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ABSTRACT

Contained herein is an informal nonmathematical survey of research in multi-echelon inventory theory covering published results through 1971. An introductory section defines the term, "multi-echelon," and establishes the kinds of problems involving multi-echelon considerations. Subsequent sections provide surveys of research on deterministic and stochastic multi-echelon inventory control problems, allocation models, and multi-echelon planning and evaluation models. A final section discusses the present state of the art and suggests directions for future research. A bibliography of papers concerning multi-echelon inventory theory and applications is included.

I. INTRODUCTION

In broad terms, multi-echelon inventory theory is concerned with a variety of inventory problems involving two or more interrelated supply or production facilities. Since most actual inventory problems of significance have multi-echelon aspects, it is natural that previous research on single-facility problems be extended along this additional dimension of complexity. To fully appreciate the directions of multi-echelon research, therefore, it is useful to have a basic familiarity with analytic approaches to the single-facility problems. Surveys of this work are given by Scarf (1963 [47], Veinott (1966 [57]), Iglehart (1967 [34]), and Hochstaedter (1969 [32]). The surveys by Veinott and Iglehart also emphasize early results of multi-product and multi-activity research.

1. Structure of Multi-Echelon Inventory Systems

The most common notion of a multi-echelon inventory system is one involving a number of retail outlets (stores, facilities, installations, bases) in business to satisfy customer demands for goods and which, in turn, act as customers of higher-level wholesale activities (warehouses, depots, factories). The wholesale activities themselves may be customers of still higher-level wholesale activities or production facilities. A grocery store chain, as illustrated in Figure 1, is a familiar example of such a system. It is important to note that the system in this example pertains to a given product such as a particular kind and brand of soup. Even for the same grocery store chain, another product may have a different structure (there may be a different factory, or no regional warehouses, or a different mix of retailers, etc.).

A multi-echelon inventory system can also be portrayed as a directed network wherein the nodes represent the various activities or facilities in the system and the linkages represent flows

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of goods. If the network has at most one incoming link for each node and flows are acyclic (no loops in the network), it is called an "arborescence" or inverted tree structure. Thus, the example in Figure 1 is an arborescence. When viewed as a directed network, it is apparent that much more complex interconnected systems of facilities can exist: a retailer may obtain resupply from more than one wholesaler, or a wholesaler may procure from more than one factory, or a retailer may sometimes supply another retailer; the number of such combinations is very large. However, most of the work in multi-echelon inventory theory has been confined to arborescence structures.

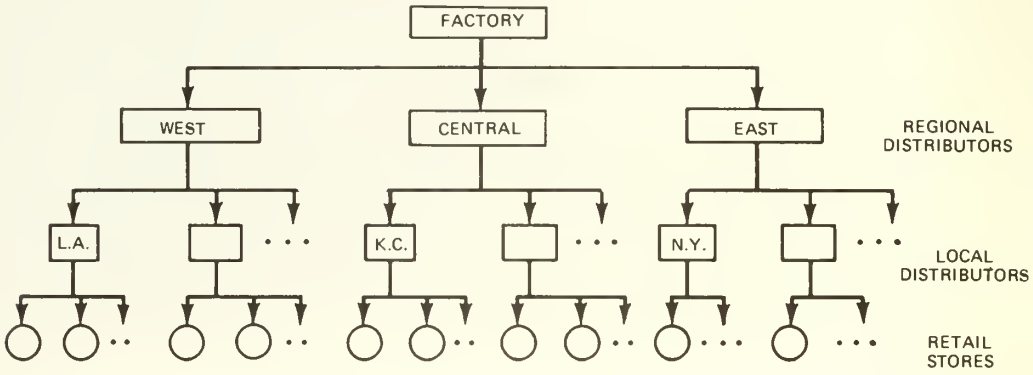

FIGURE 1. Distribution system for a grocery product

When considering arborescence-type activity structures, the various levels of the system are commonly identified as echelons and the problems investigated are multi-echelon ones. However, since this descriptor makes sense only for arborescence structures and has had even more restricting definitions in some studies, its use in the remainder of this review will be avoided except as defined and used in a particular study.

Finally, two special kinds of arborescence structures are commonly used in the literature. As illustrated in Figure 2, these are the series structure, consisting of two or more activities with each supplying only one other (lower-level) activity, and the parallel structure, consisting of a number of activities experiencing independent external demands; both cases have an implied infinite source of supply as indicated. In the series case, external demands may occur only at the lowest activity or at all activities according to the assumptions of a particular study.

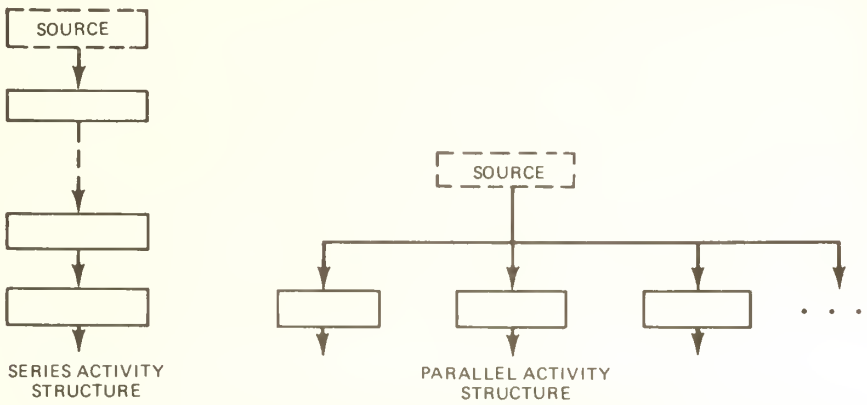

FIGURE 2. Special types of arborescent structures

2. Interpretations of Inventory "Activity."

The literature on multi-activity inventory theory contains a wide variety of labels and meanings of what is referred to here as an "activity." Most commonly, it is defined as an entity that maintains a physical stock of one or more products in order to meet specified demands and which, in turn, obtains resupply from some specified (or implied) source or sources. Other definitions, used in particular studies, expand the role of the entity to include production or repair capabilities as well as supply; thus, an activity may represent a stage of factory production, together with an in-process inventory, that satisfies requirements of other activities and which receives (orders) materials from specified source activities. Still other studies define activities as more abstract processes that are interrelated in some fashion. Due to these various meanings and nomenclatures, the more generic term "activity" will be used in this review unless otherwise explicitly defined in a particular study.

3. Multi-Activity Inventory Control Problems

Viewed in terms of a network of activities, with external demands occurring at some or all of them, the basic multi-activity inventory control problem for a given product is one of establishing rules or policies which, if followed, cause flows of the product through the network as functions of time and which satisfy a prescribed performance objective, such as, minimizing expected costs or meeting a prescribed level of customer service. The set of such policies, for any given system, usually contain *ordering policies* (resupply, procurement, production, repair), which prescribe amounts over time that each activity orders from its supplier(s), and *supply policies* (issuing, delivery, distribution, allocation, which control amounts over time that each activity ships to those activities designated as its customers. A common situation which warrants this distinction is one where there is insufficient stock (due to randomness of demand or a variety of other causes) at a particular supplier to fill all the orders it receives (as prescribed by the ordering policies of its customers) and some kind of rationing, or supply policy, is thereby required.

To solve this inventory control problem, a variety of models have been formulated which are characterized by environmental assumptions similar to those commonly applied in models of the single-activity problem. The more important of these distinguishing features are expressed by the following dichotomies:*

Deterministic—Stochastic. In a deterministic model, external demands at each designated activity are known in advance with certainty. For a stochastic model, the demands are assumed known to within a given probability distribution (or conditional distributions as used in Bayesian and other techniques).

Single-Product—Multi-Product. A single-product model deals with only one product at a time, ignoring possible interactions with other products. A multi-product model considers a number of products simultaneously in terms of at least one interrelating factor such as a budget or storage constraint.

Stationary—Nonstationary. Parameters used to define external demands are assumed to be independent of time in a stationary model, whereas they may vary over time in a nonstationary model. (The same distinction may also pertain to cost factors and other elements of the problem.)

Continuous Review—Periodic Review. In a continuous review model, opportunities to review the stock position of the system and to implement derived policies are assumed to occur continuously

*These are actually only convenient labels or catch-phrases used to categorize models roughly. The exact meanings must, in each case, be gleaned from the mathematical formulation of the model.

over time. In a periodic review model, such opportunities are assumed to exist only at discrete points in time, normally with a given periodicity.

Consumable Product—Reparable Product. In a reparable product model, some or all of the amounts issued to meet external demands are regenerated somewhere in the system in the form of items which, after undergoing repair or overhaul processes, can be reissued. In a consumable product model, all issues represent permanent losses to the system.

Backlog—No Backlog. A model assumes backlogging of demand if unsatisfied demands are retained and satisfied from later resupply. In the no-backlog (lost sales) assumption, unsatisfied demands are not retained.

In the review that follows, these descriptors are often used to identify and characterize various multi-activity inventory control and other models that have been developed. In particular, section II is concerned with deterministic inventory control models, whereas section III covers stochastic inventory control models.

4. Allocation Models

Decisions effecting the allocation or distribution of a given amount of stock to several activities in a multi-activity system represent supply policies in a restricted type of inventory control process. This class of problems also includes the one of redistributing existing stocks to provide better service. Problems of these kinds are reviewed in section IV.

5. Other Multi-Activity Problems

In addition to inventory control and allocation problems, there are several other decision, planning, or evaluation problems that have been investigated in a multi-activity environment. Solutions to these problems tend to have "one-time" applications in contrast with the everyday applications of results of the inventory control models. In section V, several such problems are identified and studies relating to their solutions are reviewed.

6. Informality of the Review

This review of work in multi-activity inventory theory is informal in the sense that the use of mathematics is avoided and there is a considerable bias of authorship with respect to selections, interpretations, and comments. The latter is particularly applicable to section VI, where impressions of the current state of the art and directions that future work might follow are presented.

II. DETERMINISTIC INVENTORY CONTROL MODELS

In the deterministic multi-activity inventory control problem, schedules of external requirements to be satisfied by each activity in the system are given as functions of time for one or more products. In this problem, many of the activities are often viewed as being various kinds of production facilities or processes for which ordering policies are interpreted as production schedules. Inventories in such systems may be held in the form of semi-finished products at various production stages or as final products at terminal facilities. Various constraints are normally imposed, such as limitations on production or transportation capacities, or interproduct dependencies of several kinds. The usual objective of models for this problem is to minimize total production, transportation, and inventory costs.

One of the early investigations of the deterministic multi-activity problem was by Evans (1958 [22]). The stationary case was considered for a general activity network with external demands given

in terms of deterministic rates at designated nodes and production occuring at other specified nodes. The formulation permits transportation and production losses and considers the production of interdependent products. Inventories are permitted, but only artificially in terms of known "stockpile rates" analogous to demand rates. Capacity limits on production and transportation are considered. With these assumptions and a corresponding cost structure, a model is developed to find flow rates through the network which minimize total expected costs per unit of time. Recognizing computation difficulties, a number of restrictions are then specified which reduces the model to a linear programming transportation problem. Finally, transport paths of the general model are restricted in such a way that results concerning multi-product systems may be derived from a two-product system and solutions to the two-product problem are given.

The first model to overcome severe limitations of early work on the problem was by Zangwill (1966 [67]). A periodic, multi-product, multi-activity deterministic model was developed that consisted of a linking together of single-activity models developed earlier (1966 [66]). The model assumes that the activities in the system are arranged in an acyclic network wherein the flow of products is unidirectional; that is, the output of any activity does not go to any other activity that directly or indirectly provides inputs to the given activity. The model is multi-product to the extent permitted by such network representations; for example, an activity with two inputs and one output can represent the combining (in a production stage) of two component products into a single output product. The model also assumes that production costs are concave, inventory costs are piecewise concave functions, demands are known in advance for a given number of time periods, excess demand is backlogged, and production/inventory times are fixed integral numbers of time periods.

Zangwill first shows that the total cost function is piecewise concave. He then shows that the function is concave on certain bounded polyhedral sets called basic sets. Using theory of concave functions, he shows that the total cost, considered as a function on a particular basic set, attains a minimum at an extreme point of the set. The union of all basic sets is proven to be the set of all feasible production schedules. The total cost function, now considered as a function of all feasible production schedules, must be minimized on some basic set, and hence, at an extreme point of some basic set. Defining the dominant set as the set of all extreme points of all basic sets, an optimal production schedule must therefore be in the dominant set. The principal result of the analysis consists of characterizations of the dominant set. For special cases of series and parallel networks, Zangwill develops dynamic programming algorithms that search the dominant set for optimal production schedules.

In a later study (1969 [68]), Zangwill shows that previous results for the case of a series activity structure without backlogging can be deduced immediately from the characterization of extreme flows in networks having exactly one source (sink). Furthermore, under the concavity assumptions on costs, there exists an optimal schedule which is an extreme flow in the associated single-source network. (An extreme flow is an extreme point of the convex set of feasible solutions for the problem and, as Zangwill previously demonstrated, has the property that any node in the network can have at most one positive input). Exploiting these results, Zangwill presents an extremely efficient algorithm that is superior to the one in Ref. [67].

Recognizing the network interpretation of the deterministic problem, Veinott (1969 [58]) pointed out that such network models are equivalent to transshipment Leontieff substitution systems and, as such, the characterization of the extreme flow follows alternatively from the characterization of the extreme points of the system's solution set. With this, Veinott expands upon the formulation of the

problem as a Leontieff substitution model with concave costs; in particular, the arrangement of activities in a general arborescence structure is considered. Veinott shows that the solution algorithm developed by Zangwill extends to this case, but that the amount of computation, dependeing linearly upon the number of wholesale facilities but to the fourth power of the number of time periods, can still be extensive. With rather severe assumptions about the cost functions, Veinott presents a simpler and more efficient solution algorithm for the general arborescence model.

After summarizing and extending previous results of the single-activity deterministic problem, Love (1968 [44]) considered the case of a number of activities arranged in series. He shows that if, in addition to concavity assumptions, per-unit ordering costs are nonincreasing over time for each activity and per-unit holding costs for each activity are always greater than or equal to those for the next activity in the series structure, then there exists a nested extreme optimal solution. He defines a nested production schedule as being one where if any activity produces in a given time period, then so does the next facility in the series structure. This result is exploited to develop a more efficient solution algorithm. Love next considers the stationary case where requirements and costs are the same in each time period. He shows that under mild conditions there is a periodic optimal schedule; that is, the amount produced at a given activity is repeated a fixed number of periods later. An algorithm is presented for computing the periodic optimal schedule for most cases of interest.

As a continuation of the Zangwill-Veinott approach, Kalymon (1970 [39]) developed a decomposition algorithm which is computationally feasible for arborescence structures that were previously too large to solve. The model assumes that holding costs are linear and that the production costs, at all but the final product facilities, are linear with a set-up charge. In the model, the amount of computation is shown to increase exponentially with the number of facilities having followers, but only linearly with final product facilities.

A deterministic, periodic, multi-product, multi-activity production/inventory problem was considered by Von Lanzenauer (1970 [60]); this work represents a generalization of previous studies by Elmaghraby-Ginsberg (1964 [21]) and Young (1967 [63]). In this study, all costs (production, holding, shortage) are assumed linear with set-up (production conversion) costs for production for each activity and product. For each product, a sequence of the activities is specified to indicate the technological ordering of production stages for the product. Each activity can process only one product in each time period. The problem is to determine the sequence of production at all activities as well as the lot size and number of lots for each product such that total costs are minimized. For this problem, a mixed bivalent linear programming model is constructed. It is noted that problems of realistic size cannot be solved by currently available integer programming algorithms.

III. STOCHASTIC INVENTORY CONTROL MODELS

In the single-activity inventory control problem with stochastic demands, a simple kind of ordering policy, identified as an (s, S)-type policy, has been subject to intensive analysis. For this policy, an order is placed to raise available stocks (on-hand plus due-in) up to S (stockage objective) whenever the available stock becomes less than or equal to s (reorder point); for dynamic models, the values for s and S may change over time. The policy parameters, s and S, are commonly referred to as "critical levels."

The (s, S)-type ordering policy has also played a major role in multiactivity research. Although a number of techniques previously used for single-activity problems have been extended to the multi-

activity problem, the use of (s, S)-type policies represents a common denominator. Some of the multi-activity research has been concerned with finding conditions under which these policies, when applied to individual activities, are optimal from a system point of view. Other studies investigate system behavior when such policies are assumed and methods are sought for assigning values to s and S at each activity such that some system performance criterion is satisfied. In general, whenever the use of (s, S)-type policies can be justified, the multi-activity inventory control problem becomes more computationally tractable; otherwise, as several of the studies have demonstrated, the amount of computation required to find optimal solutions becomes prohibitive for problems of realistic size.

For convenience, the more important techniques used to analyze the multi-activity inventory control problem are labeled as follows, with the authors of initial or early papers being indicated in each case:*

Expected Cost Analysis (Berman-Clark)
Stationary Process Analysis (Love)
Dynamic Programming (Clark-Scarf)
Dynamic Process Analysis (Bessler-Veinott)
Network Theory (Connors-Zangwill)

In most of these approaches, several papers have appeared which extend or refine the initial development; these are reviewed below in approximate chronological order.

1. Expected Cost Models

In expected cost models, the form of the inventory policies is expressed in terms of a few variables; normally, (s, S)-type policies are assumed with s and S considered as unknowns. Expressions for expected costs per unit of time are then constructed as functions of the policy variables. Values for the policy variables that minimize the expected average costs are found by calculus or difference equation methods. Such models are further characterized as being stationary and usually stochastic; either continuous or periodic review may be assumed. The first analytic work on inventory problems used this approach; in fact, the famous Wilson lot size formula is the solution of such a model (with deterministic demand).

One of the early attempts to extend this approach to the multi-activity inventory problem was by Berman-Clark (1955 [5]). This work treated a single product in a two-level activity structure (several bases supplied by a depot) in context with three main procurement practices: life-of-type, where sufficient amounts are purchased all at once to satisfy all expected future demands; periodic procurement, where a system purchase is made at regular time intervals in sufficient quantities to last throughout the period; and open-end contracting, where an order on a manufacturer is placed each time the system-wide stocks reach a minimum (critical) level. For each type of procurement, expressions for expected average costs, as functions of (s, S)-type policy variables, are derived which include representations of the following features: random external demands at bases; repair or no repair (reparable or consumable items); two kinds of resupply time for bases, normal and expedite; transshipment among bases; obsolescence probabilities; random leadtime and repair cycles; and the usual holding, shortage, and ordering costs, all linear. From these cost functions, expressions for minimizing

*Other techniques that have been used, or are potentially applicable, include stochastic linear programming, servomechanism theory, and simulation.

values of the policy variables are obtained. For bases, the critical levels are functions of Lagrangian multipliers which are used to control distribution among the activities in the system.

Hadley-Whitin (1961 [28]) also used the expected cost approach to analyze the inventory problem for a low-demand item in a single-level system of parallel activities (depots). Demands at each depot are assumed to be independent, stationary Poisson distributions. Procurement lead time is assumed to be constant as well as the time required for either of two available modes of redistribution among the depots. Items are ordered one at a time from an external source as demands occur; therefore, the total system stock remains constant over time. It is assumed that redistribution to a depot will not occur until there is a backorder. With these assumptions and linear functions for inventory costs, decision rules are developed for allocation of new procurement, redistribution of stocks among the depots, and for determining depot stockage objectives in order to minimize expected costs resulting from system and depot stockouts, cost of redistribution, and costs of transportation from the source. When more than a single unit is on order from the external source, the allocation of each unit ready to be delivered is determined by the solution of a dynamic programming problem. The assumption that demands are replaced one-for-one by procurement and the form of the redistribution policy confine the model to low demand items.

In a later study (1963 [29]), Hadley-Whitin used the same general approach to consider the case of higher demand items. This time, it is assumed that the system as a whole uses an (s, S)-type policy and that redistribution is considered whenever a depot's stock falls to critical levels that are set by external criteria. Furthermore, only that depot triggering the redistribution decision is considered as a receiver each time. All other assumptions for this model are essentially the same as for the previous one. Again, cost minimizing expressions are obtained for determining stockage objectives at the depots, values for the system procurement policy, and sources and amounts for redistribution decisions. A dynamic programming algorithm is presented for allocating system procurements to the depots upon receipt from the external source.

The same problem (a number of parallel activities in a single-level system subject to centralized control) was also investigated by Gross (1963 [26]). In this study, however, attention is confined to a single time period and no a priori assumptions are made concerning the form of stockage and redistribution policies. The objective is to find amounts to order and/or to redistribute at the beginning of the period such that total expected costs during the time period are minimized.* In the model, all costs (ordering, holding, shortage, transshipment) are assumed linear without set-up costs. Deliveries into the system and transshipments occur at the start of the period (zero procurement and transshipment times). External demand may be given by any continuous density function which may differ by location. Under these assumptions, a total cost function is first formulated for the case of two locations. From this, minimizing values for ordering and transshipment amounts are derived which are dependent upon the initial stock posture at each activity and the system as a whole. An iterative procedure is then given to generalize the results to an arbitrary number of locations.

The problem formulated by Gross was also considered by Krishnan-Rao (1965 [41]), but with one important difference. Instead of allowing transshipments to occur at the start of the period in anticipa-

*It should be noted that confinement of attention to a single time period avoids implicit assumptions, which are not always valid, concerning the stationary properties of the inventory process in infinite horizon expected cost models. Thus, this model can be considered as being truly optimal for the stated problem, whereas the other expected cost models are generally only approximations.

tion of expected shortages, it is assumed that shortages actually realized at the end of the period are satisfied from stocks at activities with ending surplus; thus, transshipment amounts are not considered as policy variables as in the Gross model. Because of this simplifying assumption, Krishnan-Rao obtain a simple solution to the problem of determining ordering amounts that minimize the expected costs. Numeric examples are given to illustrate results of the model.

2. Stationary Process Analysis

If nonstationary aspects of inventory problems are removed, then techniques based upon Markov processes and elements of renewal or queuing theory become useful. Here, a fixed ordering policy of a simple form (usually an (s, S)-type policy) is chosen for which the inventory level over time becomes a particular stochastic process. The principal problem then is to find a stationary distribution of the process which, if it exists, will be a function of the policy used and of the demand distribution, but not of any costs that might be involved. However, a cost structure can be imposed upon the process in expressions representing average expected cost per time unit. These may then be minimized with respect to the several parameters that characterize the policy.*

One of the first investigations of a multi-activity inventory control problem using a stationary analysis method was by Love (1967 [43]). In this study, a two activity series system is considered, with both activities using continuous review $(S-1, S)$ policies (such policies are also commonly called one-for-one ordering policies). External demand, occuring only at the lower activity, is assumed to be given by a Poisson distribution. Resupply time at both activities is exponentially distributed. Per unit shortage costs are assumed to be linear according to age of the back-order. Holding costs are linear and reorder costs are not considered since ordering is one-for-one as demands occur. Under these assumptions, expressions are obtained for expected total waiting time (e.g., backorder days) and stock on-hand per unit of time. Love then demonstrates convexity properties of the expected total system cost as functions of the stockage objectives and, as a result, the existence of optimal stockage objectives (values for S at each activity). He then establishes bounds on the minimizing values and gives an algorithm for determining the optimal policies.

The inventory control problem for a repairable item in a two-level supply/repair system was investigated by Rosenman-Hockstra (1964 [46]) using a stationary process approach. In this study, it is assumed that items can be repaired locally (area facilities) or centrally (NICP—National Inventory Control Point) according to given rates and that losses to the system are negligible. It is further assumed that external demands are Poisson distributed, that lower-level facilities use continuous review, one-for-one $(S-1, S)$ ordering policies, and that replenishment times and repair cycles are given constants. Under these assumptions, a cost-free model is developed with the objective of distributing a given system stock among the various activities in order to minimize total expected customer waiting time.

In the model, stationary distributions for number of items in the NICP repair cycle and net (serviceable) stock at the NICP are developed, from which an expression for average delay in satisfying demands from lower activities is derived. Similarly, distributions for repair cycle and stock on-hand at the area facilities are given, from which the average number of backorders are obtained and summed

*Although the expected cost method is similar to this procedure, it does not include a formal analysis of the stationary properties of the underlying stochastic process; instead, implicit assumptions are made in this regard which are not always valid.

over all areas to find the average customer waiting time. Using these results, a marginal value method is used to distribute system stock in order to minimize total customer waiting time.

In addition to the model itself, the paper presents actuarial and exponential smoothing techniques for demand forecasting, and discusses institutional methods and procedures for implementing the results.

Sherbrooke (1968 [48]) also used the stationary process approach to analyze the multi-product problem involving inventories of recoverable (repairable) items in a system of parallel activities (bases) supported by a higher level activity (depot). In the model, a Bayesian procedure is used for estimating external demands (occurring only at base level), with the prior distribution being a gamma function and the underlying demand being a logarithmic Poisson process. The derivation of these distributions is discussed in some detail. Although the expression of demand as a Bayesian process is separable from the multi-activity and other structural aspects of the model, the use of the indicated distributions enables the derivation of a distribution of demand observed by the depot that provides a decomposition of the problem.

The model assumes $(S-1, S)$ continuous review policies at lower activities and since all stocks are conserved (being always repairable), there is no system reordering after initial stocks are established. The primary objective of the model is to establish stockage objectives (values for S) at each activity which minimize the sum of expected backorders on all recoverable items at the lower installations for a given budgetary constraint. Each recoverable item may be repaired at base level and/or at depot level according to a specified ratio. Economies of scale are not considered and lateral resupply (redistribution) is assumed to not occur. Relative backorder costs may be included in terms of essentialities by base and by item. Repair and shipping times are assumed random (implicitly in the assumed form of the demand distributions). Under these assumptions and objective function, a five step procedure is given for finding optimal solutions. First, using a previously developed expression for expected number of base backorders, the average delay per demand against the depot is found for each item as a function of depot stock. Second, for each level of depot stock and each base, expected base backorders are calculated as a function of the base stock. Third, for each level of depot stock, an allocation to the bases is made which minimizes the total expected backorders; this is done by a marginal analysis method. Fourth, the minimum expected system backorders are found as a function of total system stock (bases plus depot). Finally, the multi-item aspect is considered by the use of a marginal value method to allocate a given investment across items: each additional increment of investment is assigned to that item for which the largest reduction in expected system backorders will result.

The mathematical justification for this computational procedure is then given wherein the objective function is also generalized to allow constraints upon the expected number of backorders at each base. For the generalized objective function, an integer programing problem results which can be reformulated with Lagrange multipliers and is separable by item. However, computational difficulties arise if the Lagrange multipliers differ by base; this occurs whenever separate backorder constraints are given by base. If they are all assumed equal, the marginal analysis method used in the computation procedure provides a good approximation to the optimal solution, particularly if the number of items is large enough.

Although the model assumes stationarity, it is suggested that periodic recomputations be made whenever systemic changes in the support or demand structure become known. The paper concludes with two generalizations of the model: the use of a Bayesian procedure for estimating the ratio of base

versus depot repair, and the inclusion of condemnation (system losses) in the model formulation.

Confining attention to stationary properties of the inventory process, Simon (1971 [50]) refined and extended corresponding portions of the Sherbrooke model. As before, a parallel structure of bases supported by a depot is considered where repair of a failed item can occur at base and/or depot level. The model allows system losses to occur at a specified rate; a totally reparable or totally consumable item can be considered according to the value of this condemnation rate. For bases, an $(S-1, S)$ replenishment policy is assumed; at the depot, an (s, S) policy is used. Repair times, shipping times, and procurement lead-time are all assumed to be deterministic and independent. External demands are assumed to be generated by independent Poisson processes with given rates. With these assumptions, Simon investigates the stationary properties of a given item in the multi-activity structure. For each base, he obtains exact expressions for the steady-state number of backorders and proves that the number of units in repair is stationary and Poisson distributed. Similar results are obtained for the depot. For a consumable item, exact and approximate expressions are derived for the stationary expected depot response time to a base demand; the approximate expression is analogous to one used by Sherbrooke.

3. Dynamic Programming

The mathematical technique most often used in formal inventory theory is the functional equation approach of dynamic programming. In this method, periodic review is assumed and optimal policies for a given time period are found from a cost function that normally consists of three parts: (a) costs of ordering, if any, (b) expected (period) costs which are independent of the ordering policy, and (c) expected future costs (from the next period on) which do depend upon amounts ordered. Under suitable restrictions upon cost and demand processes, simple types of cost minimizing policies exist and may be calculated by a recursive process which starts at some future time period and works backwards in time.

As a continuation of the classic dynamic programming approach used in single-activity problems, Clark (1958 [13]) and Clark-Scarf (1960 [16]) formulated and solved the problem of a single product carried at an arbitrary number of activities arranged in a series structure. In the model, external demand can be stochastic, dynamic, and can occur at any arbitrary mix of the included activities. The model permits economies of scale (fixed reorder costs) only at the highest activity in the series structure. At lower activities, ordering (shipping) costs are assumed linear as functions of amount ordered. The model assumes periodic review and expected period costs (holding and shortage) are assumed convex. Shipping times are constants but can be an arbitrary number of periods for each activity. Backlogging is assumed at all activities.

The essential innovation of this model is the interpretation of the inventory system as a nested set of echelons rather than as individual activities. The model associates, with each activity, an echelon consisting of all stock in the system at that activity and below, including all on-hand and in-transit (due-in) amounts. With this interpretation, the multi-state variable problem for the system as a whole (i.e., the on-hand and due-in amounts at each activity are separate state variables in the total cost function) can be decomposed into a set of interconnected one-state variable problems, one for each echelon in the system. Each of these problems can then be solved, by the usual single-activity technique, for critical levels constituting the optimal ordering policy.

The set of one-state variable problems are interconnected by "implied shortage costs" generated at each echelon (except the highest one) and passed on (included in the cost function) to the next higher

echelon. Thus, the optimal policy is first established for the lowest echelon, from which implied shortage costs are obtained. These costs are then included in the cost function for the next higher echelon for which the process is repeated. The procedure continues up the echelon structure until optimal policies are obtained for each echelon. The ordering policies determined in this fashion take the form of periodic $(S-1, S)$ policies at lower echelons and an (s, S)-type policy at the highest echelon. The issuing policy is that each activity satisfies as much of an order (demand) as may be permitted by available stocks in each instance.

In Ref. [16], a brief analysis of the parallel echelon structure is given. It is shown that, in general, the problem cannot be broken down into a set of single-activity problems. However, a solution of this character is given under the assumption that the stock positions at the several lower installations is continually "in balance," i.e., that the stock position at each activity is not significantly over or under that of the other activities relative to expected costs. A concluding short section of this paper discusses extensions of the theory to serial repair and production processes.

In a subsequent paper (1962 [17]), Clark-Scarf considered the case of a fixed recorder cost at lower installations with all other elements of the problem the same as before. In this case, it was found that the optimal solution could not be broken down into a sequence of single-state variable problems, but that the cost function for the optimal solution could be bounded (above and below) by such calculations. Thus, critical levels for each activity and time period could be found with associated costs which bracket those of the optimal solution. Furthermore, it is demonstrated that if the policies (critical levels) for the upper bound were used, then the actual costs incurred would be less than or equal to those of the upper bound.

A more comprehensive multi-activity inventory control model based upon the Clark-Scarf approach was developed (in 1962) by Clark-Metcalfe; although the model itself was not documented (except in the form of a computer program), example results are given in Ref. [14]. This model incorporated features which, although not proven optimal, provides more realistic representations of practical inventory problems. The model permits stochastic, dynamic external demands at any activity in a general arborescence structure, recovery through repair (the treatment of this aspect of the model was later shown to be optimal [9]), deterministic time lags (procurement, resupply, repair cycles) of arbitrary length, multi-stage production with inventories permitted between stages, and a form of continuous stock review. The cost structure of the model is analogous to that of the Clark-Scarf model. For portions of the activity structure consisting of activities in series, the implied shortage cost functions of the Clark-Scarf method are used to interconnect echelons. For portions consisting of parallel activities, a shortage cost function is built up from the cost functions of the lower activities to provide echelon interconnections (see Ref. [45] for a description of this procedure). Outputs of the model consist of values for a single critical level at each lower activity and values for an (s, S)-type policy at the highest activity; a supply policy is derived from the implied shortage cost functions. In conjunction with the decision model, a simulation model was constructed to provide realizations of system performance while operating under the ordering and supply policies generated by the analytic model.

Continuing the Clark-Scarf formulation, Hochstaedter (1970 [31]) considered the case of activities in parallel with a common supplier. The cost and demand structure was assumed the same as the Clark-Scarf model, but with fixed reorder costs permitted at the lower activities as well as at the higher echelon. As in Ref [17], upper and lower bounds are established for the optimal system costs, with each set of bounds yielding (s, S)-type ordering policies for each activity. The bounds are established by first computing ordering policies for each lower activity using the standard single-activity formula-

tion. It is shown that the optimal system costs are then bracketed by the sum of the lower activity costs plus a function of system stock alone. For the lower bound, the function is given by the costs obtained by viewing the system as a single installation faced with a demand equal to the sum of the lower installation demands. The upper bound is given by the same cost function, but with period costs (for the higher echelon) being augmented by an implied shortage cost derived from the lower installation computations; these costs are analogous to those previously described, but assume a different functional form. As in Ref. [17], Hochstaedter suggests the use of levels associated with the upper bound as an approximation to the optimal policy. He also derives an expression for the difference between the upper and lower cost functions which is useful in assessing the accuracy of the approximation.

The extension of the Clark-Scarf series multi-activity model to include combined ordering and disposal policies was accomplished by Fukuda (1961 [25]). In general, disposal policies determine conditions under which excess stock is removed from inventory at some cost which may be negative indicating a revenue from the disposal. Addressing first the single-activity case, Fukuda assumes that the decision to be made at the beginning of each period is to either order, dispose, or neither. With this assumption and the usual cost structure, he determines optimal ordering-disposal policies in terms of critical levels for each time period. These results extend directly to the series echelon structure wherein critical levels are obtained for each activity and time period. In this case, implied shortage costs for disposal are derived which are analogous to those for ordering in the Clark-Scarf model; such costs apply whenever a higher echelon wants to "sell" some excess stock, but cannot because the stock is physically located at some lower level. The implied shortage costs for disposal are passed downward (included in the cost function of the next lower echelon) in contrast to the implied shortage costs for ordering which are passed upward in the structure.

In previous paper, Scarf and Fukuda developed, respectively, Bayesian and maximum likelihood policies for the single-activity problem. Fukuda (1960[24]) extended these results, combined with the disposal policies of Ref. [25], to the Clark-Scarf series model. For the Bayesian policy, Scarf had shown that the single-activity problem could be reduced to a one-state variable problem if all cost functions are linear and if the demand distributions as well as the prior distribution of the demand parameters are members of the gamma family. Fukuda now shows that this result extends directly to the series multi-activity case; namely, that the solution of an interconnected sequence of one-state variable problems provides optimal Bayes ordering policies for the series structure. Furthermore, the extension holds up in the situation where each activity in the system experiences external demand; thus, Bayes policies are determined for various serial repair and production processes. Finally, Fukuda extends the results to the disposal case, resulting in a Bayesian ordering and disposal policy for the series situation. Analogous results are obtained for the maximum likelihood approach, wherein demand is assumed to be given by a gamma distribution with an unknown parameter and critical levels (ordering and disposal) are obtained for each activity and time period as linear functions of the unknown parameter which are then evaluated by expressions based upon past demand observations.

Veinott [57] used a convexity theorem of Karush [40] to yield a much simpler proof of optimality for the Clark-Scarf model.* The theorem was also applied by Iglehart-Morey (1971 [36]), in the case of two facilities in series, to obtain optimal ordering and shipping policies that take into account the accuracy

* Veinott (1971 [59]), using a network approach, also gave a simple solution to a one period problem of a series structure with zero resupply times.

of demand forecasts; otherwise, the model is the same as in Ref. [16]. In this model, it is assumed that demand forecasts are made each period which may differ from the true demand. The cost function for the model includes conditional expectations with respect to demand given the forecast. As results, critical levels are obtained that are analogous to those of the Clark-Scarf model except that the levels for the lower activity are expressed as a function of the forecast.

Zacks (1970 [64] formulated a Bayesian model of the two-echelon parallel activity structure, assuming that Poisson-distributed demands occur at the lower activities and that the prior distribution of the Poisson parameters is a gamma distribution. He assumes no transshipment among lower activities and no return of stock to the higher activity. In the cost structure of the model, linear holding and shortage costs are assumed, but no ordering or shipping costs are included. With these assumptions, a multi-state variable dynamic programing solution procedure is developed. The optimal Bayesian policies are expressed as functions of on-hand and due-in amounts at the various activities; the analysis does not provide simple policies in the form of critical levels and rules for their use.

In a later paper (1970 [65]), Zacks reconsiders the model, this time allowing unrequired stock at lower activities to be returned to the higher facility, linear ordering costs (but no fixed component) at the higher facility, and no shipping costs between echelons. The same general Bayesian approach as in the earlier model is used, and all other assumptions are the same as before. Again, a dynamic programing formulation is presented, from which the form of the optimal policies is derived. The principle result of the analysis is that the optimal ordering policy of the lower activities is obtained by solving an integer convex programing problem with linear constraints. The ordering policy for the higher activity is expressed as the minimizing value of a recursively defined function; ordering amounts depend upon stock on-hand plus due-in and upon past demand observations.

The case of activities in a series structure, where each activity has a fixed ordering cost in addition to the usual inventory costs was investigated by Williams (1971 [61]). Using the Clark-Scarf definition of echelon, a multi-state variable dynamic programing model is developed for both the backlog and lost sales case. For the backlogging model, the assignment of shortage costs as a function of the age of the backorder is considered, but it is noted that this increases the number of state variables of the model. The special case where a product unit progresses through all activities in the series structure within the same time period is considered in the form of a submodel. For the submodel, calculation procedures are presented which are considerably more efficient than those of the more general model.

4. Dynamic Process Analysis*

In previous papers, Veinott developed a technique for analyzing inventory problems, first for the single-product, single-activity case and then for the multi-product, single-activity problem. Veinott's methods represent a departure from the traditional functional equation approach of dynamic programming, being based instead upon a direct analysis of the underlying stochastic process to obtain the form of optimal policies and to investigate their properties. These results were extended by Bessler-Veinott (1966 [7]) to the multi-activity inventory problem.

In the first section of this paper, the correspondence between the multi-activity and multi-product, single activity problems is established. In general, when the stock at each facility is viewed as a product, results of the multi-product, single-activity problem have corresponding interpretations for the single-

*This description for the Veinott technique was coined for purposes of this review.

product, multi-facility case. The principle result is that if nonstationary aspects are suitably restricted and the initial stock is small enough, the optimal policy in each time period is to order up to a critical stock level at each facility, where the critical levels minimize a one-period cost function which includes the usual inventory costs except for fixed reorder costs.

In the next section, the arrangement of the activities in an arborescence structure is considered. It is assumed that the demands during each period at each activity are satisfied by available stocks at the facility, with excess demands being immediately transmitted to its supplier for possible satisfaction. Excess demands are successively passed up, with backlogging occuring only at the top supplier. With this supply policy, the one-period cost function is established from which the optimal stock levels are derived. In general, the optimal policy is obtained by solving N n-dimensional minimization problems, where N is the number of periods and n is the number of activities.

Having obtained the one-period cost function, several effects of parameter variations are investigated, supported by numeric calculations for a system consisting of three parallel activities supplied by one higher-level activity. Various properties of the shortage cost function and the one-period cost function are also established for use in the subsequent development.

Next, bounds are established for the optimal stock levels. The upper bound for a particular activity is based upon the assumption that there is no stock at any facility supplying or supplied by that facility. The lower bound is then calculated under the assumption that each facility supplied by the facility stocks its upper bound and the supplying facility has a very large stock. An iterative procedure is then given to sharpen these bounds, although it is suggested that, in practice, it may not be worthwhile proceeding beyond the initial values. Finally, an algorithm is presented for computing approximations to the optimal levels based upon the values for the lower bounds; it is noted that the approximations are often optimal for the series activity structure.

In the last two sections of the paper, several relations among facilities having a common supplier are established and simplifications resulting when there are symmetries in the supply structure are discussed. In particular, it is proven that if two facilities having a common direct supplier "look alike" and if certain convexity assumptions are imposed, then it is optimal to stock the same quantities at both facilities.

In the Bessler-Veinott model, it was necessary to place restrictions upon starting stocks in order that optimal policies be given by minimizing one-period cost functions. This restriction was removed by Ignall-Veinott (1969 [37]) for activity networks having a "nested" property and a specified supply policy; this paper also established the characterization of policies satisfying the one-period minimization criterion as being "myopic." The model presented assumes that at the beginning of a period, stocks are ordered for each activity at a common unit cost. A supply policy is given in terms of a procedure for allocating the starting stocks to satisfy demands at the facilities. This procedure is based upon identifications of groups of activities as being suppliers of other groups of activities. The model assumes that stocks at one facility may satisfy shortages at any facility providing that they are replaced from exogenous sources at the beginning of the next period. With these assumptions and the cost structure of the previous model, it is shown that the myopic policy is optimal for any initial stock posture. This result, established first for the stationary case, is then extended to the case where there is a delivery lag and to cases of nonstationary costs and demand distributions.

5. Network Theory

Although multi-activity inventory systems have long been recognized in terms of a network of activities with flows of material occurring among the nodes, the first application of network theory for the stochastic inventory control problem was by Connors-Zangwill. In principle, this approach is an extension of the network analysis of the deterministic problem.

In the Connors-Zangwill model (1971 [18]), requirements or availabilities at the nodes are allowed to be discrete random variables with known conditional probability distributions. For a given number of periods and linear objective criterion, the problem is to calculate the network flow that minimizes expected cost. To handle this problem, special kinds of networks, called "replicating networks," are defined which contain two special types of nodes in addition to the normal ones: replicating nodes and collating nodes. A replicating node has a single input arc and several output arcs, such that the flow on each output arc is identical with that on the input arc; a collating node has the opposite property. After formulating a problem in terms of a replicating network it is transformed into an equivalent replicating network having only one source and one destination. For the transformed network, an efficient computation algorithm is presented which is analogous to the standard min-cost/max flow algorithm for deterministic models. It is shown that although this problem can be solved by linear programing under uncertainty, the network approach is much more efficient for large problems. The method and computation procedure are illustrated by a two-period production/inventory problem with backlogging.

IV. ALLOCATION MODELS

The problem of allocating or distributing a given amount of system stock to the various activities in a multi-activity system may be viewed as a restricted type of multi-activity inventory control problem. If such allocations include existing stock postures at the activities, the problem becomes one of determining redistributions in order to satisfy a system cost or performance objective. Such redistribution problems naturally occur when stocks at the various activities get "out-of-balance" or maldistributed because of the random characteristics of demand or previously unknown shifts in demand patterns.

Studies of the allocation problem are characterized by the assumptions that are made concerning the problem to be solved and by the analytic technique that is used. The particular models reviewed below are representative of the various kinds of problem formulations and solution procedures that have been developed.

One of the earliest investigations of the allocation problem was performed by Simpson (1959 [52]). He assumes that there are a number of activities (warehouses) faced with independent external random demands and that the system, as a whole, receives a given amount of stock for a particular item which is to last (satisfy demands) for a fixed length of time. Emergency replenishment is allowed whenever the stock at a warehouse is reduced to a previously established trigger level; it is assumed that a fixed cost penalty is charged each time this happens and that no distinctions are made as to the source of the replenishment (external or internal). The problem then is to allocate the given quantity to the warehouses to satisfy given performance criteria. Two criteria are considered: system cost minimization and minimization of weighted number of unsatisfied demands.

Under these assumptions (and others that are not explicitly stated), simple allocation rules are obtained for the two performance criteria. For the cost minimization criterion, the rule states that the probability (weighted by emergency resupply penalty) of demand being equal to the quantity allocated minus the emergency trigger level be equalized across all warehouses. For the second criterion, where

weights are specified across warehouses for lost sales, the rule states that the probability (times the weighting factor) of demand exceeding the quantity allocated be equalized across the various warehouses.

A redistribution problem involving a one-time decision to relocate stocks among several user locations was investigated by Allen (1958 [1]). A model is constructed with the objective of minimizing transportation costs plus shortage costs at the end of a defined period of time. It is assumed that transportation costs, for any pair of locations, are proportionate to the amount moved between the pair. With given assumptions about the demand distributions, various properties of the solution to the problem are established which permit the use of a simple iterative computation method. With less general assumptions about unit transportation costs, even simpler methods of solving the redistribution problem are obtained. In particular, the cases of zero costs and costs independent of origin and destination are examined.

Allen continued the study of this problem (1961 [2]), but this time with a fixed set-up charge for each shipment from one location to another. Also, a normal distribution of demand at each activity is assumed, and only the case where per-unit transportation costs are all zero is considered. With these assumptions, an expression for total system cost of redistribution is obtained which is transformed into a function dependent only upon the total shipments from or to an acitivty. From this, total system cost is derived as a function of normalized stock positions before and after redistribution and of a 0–1 variable identifying activities that actually ship or receive material. A method for minimizing this function is then derived and a computing procedure is suggested. Further development of the computation procedure is given in Ref. [3].

The classical dynamic programing formulation of the inventory control problem is used by Iglehart-Lalchandani (1967 [35]) in a periodic review allocation model. A system of two activities is considered, each experiencing external random demands that are not necessarily independent. Period costs (expected holding and shortage costs) are assumed to be convex; shipping costs are linear without a fixed component. A critical final assumption is that the sum of the inventory levels at the two installations after delivery of any allocated stock must be less than a given fixed amount. Thus, the problem is one of periodically allocating a fixed total system stock to the two activities. Under these assumptions, it is found that the optimal (least cost) policy is given by critical levels for each activity and time period, and simple rules are given for their use which depend upon the respective stock levels at the two activities. It is stated that an extension of the problem to more than two installations would require additional techniques to those used in this study.

V. PLANNING AND EVALUATION MODELS

In contrast with inventory control models that are directed towards managing the everyday flow of material in a multi-activity inventory system, a number of models have been developed for use in various kinds of planning or evaluation problems. As examples of such problems, it may be desired to choose particular activities in a multi-activity system which should carry each product in a multi-product inventory in order to minimize overall costs, or evaluations are desired of the relative merits of several different postulated multi-activity inventory control policies, or locations are to be found in an existing multi-activity system at which stocks are to be positioned to meet possible one-time requirements. Models that have been developed for these and other problems of similar nature are reviewed in this section.

1. Inventory Positioning Models

Given an inventory system consisting of a number of interconnected activities that collectively carry a variety of products, it can well happen that the mix of external demands in terms of where they occur, as well as identifications of external sources, can vary from one product to another. Also, even though demands can occur at a given activity, it may not be economical to carry inventories there in view of alternative supply opportunities elsewhere in the system. An important problem in this situation is one of determining, for each product, the particular activities at which inventories should be carried such that total system costs are minimized or some other system performance objective is satisfied. Different aspects of this problem have been investigated in the studies reviewed below.

One of the earliest investigations of this kind of problem was a study by Simpson (1953 [51]). This work was concerned with a multi-stage manufacturing process where in-process inventories may be carried. The general problem is to determine stages of the process that should carry in-process inventories and amounts to be carried at each such stage in order to achieve a desired level of operating efficiency. To solve this problem, Simpson formulates an expected cost model which assumes an ordering mechanism wherein each unit withdrawn from any stage is immediately reordered from the previous (next higher) stage. Also, it is assumed that for included inventories, stockage objectives are set in advance to provide specified levels of service; thus, assessment of shortage costs are avoided. It is shown that the domain of the cost function under these assumptions is a convex polyhedron and that the minimum is attained at a vertex of the polyhedron; each vertex corresponds to a particular combination of production stages that should carry inventories. It is suggested that the cost function be evaluated at each vertex in order to select the minimum cost solution.

Hansmann (1959 [30]) continued work on the problem presented by Simpson, but under less restricting assumptions. He permits periodic ordering for each inventory, where ordering decisions are based upon the actual inventory level, quantities due-in, and forecasts of future demands (sales). Demand is assumed to be stationary and given by normal distributions. Sales revenues and inventory holding costs are considered, but a functional relationship between average delivery time and expected demands is used in lieu of shortage costs. System optimization is therefore defined as the maximization of the difference between sales revenues and inventory carrying costs. To solve this problem, Hannsmann first derives expressions for expected overages and shortages as functions of a target (critical) level for a particular product and activity. These results are used to derive an expression for expected profit (revenue minus inventory cost) as a function of delivery time which is then maximized to find the optimal delivery time from which the optimal ordering policy is found. These results are then related to the multi-activity situation, where delivery times for each activity are dependent upon the existence of stockages at higher activities. An expected profit function is developed containing expected delivery times as variables. Again, maximizing yields optimal delivery times and, hence, optimal ordering policies; if the resulting target level is zero for a given activity, then no inventory is carried there. Finally, the same approach is extended to the multi-product, multi-activity problem. A dynamic programing procedure is used to find delivery times and corresponding positionings of inventories that maximize expected profit.

In another study, Brown-Silverman-Perlman (1971 [10]) consider a parallel system of activities (aircraft carriers) supported by a forward supply point where all included activities carry all products of interest. This system, in turn, is supported by a higher-level system which is abstractly represented in terms of alternative resupply processes and subprocesses (these processes may be viewed as "activities" in the broad sense). It is noted that the length of time required to resupply the lower system is dependent

upon which combination of higher-level processes are incurred; this length of time, in turn, affects the amounts and costs of inventories required in the lower system. Since each higher-level resupply activity has associated costs, the problem is one of selecting a combination of the resupply activities, from among those available, for which the sum of inventory costs and resupply process costs are minimized.

Inventory costs in the lower system are determined by repeated use of a single-activity model by Brown-Corcoran-Lloyd [9]; this is an extension of the classical dynamic programming model to include reparable items. Application of the model to the lower activities (all assumed to be the same) yields critical levels from which a demand probability distribution against the higher activity (forward supply point) is derived and approximated by a Poisson distribution. The model is then applied to the higher stock point considered as a single activity. It is noted that this procedure results in overstated inventory costs, but the analysis proceeds under these assumptions.

Next, the higher-level resupply system is considered in terms of alternative subprocesses, each having an associated response time and level of expenditure (in general, the level of expenditure increases as response time decreases). The total system cost is then given as the sum of costs for selected processes and resulting inventory costs of the lower system which depend upon the response times of the selected processes. The optimization problem becomes a 0–1 integer programing problem which is solved by a branch-and-bound algorithm. The model is applied to each product independently; no account is made of possible economies of scale in the resupply process expenditures.

A study by Pincus (1971 [45]) deals directly with the inventory positioning problem where all activities in the multi-activity system represent different stockage facilities. In this study, a basic multi-activity structure is assumed, from which a substructure is to be selected for each product in order to minimize expected costs over all products and facilities. For each possible substructure, inventory costs will result for each product and each included activity will have various fixed (capital expenditures) and recurring (operating) costs. Based upon a matrix of such costs for all possible activity substructures and all products, a 0–1 integer programing problem is formulated, the solution yields a substructure for each product such that total inventory and facility costs are minimized. A branch-and-bound algorithm is presented to solve this problem.

As part of the inputs to this model, it is necessary to derive expected inventory costs for a given product in a given multi-activity substructure. Although it is pointed out that any one of the several available multi-activity inventory control models can be used to obtain these costs, a model based upon the Clark-Scarf [16] approach was formulated for purposes of the study. The use of this model is illustrated by numeric examples. Finally, an example consisting of 30 products and 10 activity substructures is presented to illustrate the use of the overall design model.

2. Repair-Level Decision Models

Many multi-activity inventory systems consist of stocks of spare parts used in repair and overhaul processes for some end article or articles. Viewing the end item as a hierarchy of component parts an important problem is one of determining, for each component in the hierarchy, whether or not to repair it when it fails; this is also known as the repair versus discard decision problem. In a multi-activity inventory system, an interrelated problem is one of deciding where to repair a component if it is to be repaired. Solutions to these problems will control the mix of inventories at each activity in the system as well as the mix of other required resources such as repair equipment and personnel. It may be noted

that this problem is analogous to the inventory positioning problem but with additional dimensions of complexity.

Although this problem has been investigated in several studies, the one by Williams (1969 [62]) is perhaps the most general. Here, a model is developed which accepts as inputs descriptions of the hierarchical structure of the end-item(s), both in terms of the physical breakdown of included parts and in terms of steps taken to isolate faults that might occur. Also input are factors describing the multi-activity repair and supply system, reliability factors for included parts, and various cost functions relating to the repair and supply operations under stationary assumptions. Based on these inputs, the model uses a dynamic programing solution method to find combinations of repair actions to be taken which minimize overall support costs. As outputs, the model specifies whether or not a component should be repaired when it fails and, if repaired, which facility in the structure should do the repair. These outputs, in turn, determine for each item the particular activities in the system that will experience demand because of the repair decisions and, hence, where inventories of the item should be established. The outputs also control the positioning of maintenance resources in an analogous fashion.

3. Contingency Support Models

An important problem in military (and some commercial) organizations is the acquisition, maintenance, and distribution of inventories in support of various kinds of postulated contingency operations. The problem has multi-activity aspects of two general kinds: (a) In the normal supply system, what augmentations of inventories are needed for anticipated contingencies and at what echelon level or levels should the contingency stocks be carried? (b) In actual contingency operations, what supply echelon structures should be constructed to best support operations? (This question is of particular importance in the Army where contingency operations often create three- or four-echelon supply systems that change dynamically in terms of structure and location). The general problem is further characterized by a practical inability (with certain exceptions) to deal with individual products; instead, the analysis for planning purposes must be based upon the consideration of product aggregates or classes of relatively homogeneous commodities.

It has long been recognized that solutions to these problems are critically dependent upon the types of contingency operations considered and upon the types and capabilities of various kinds of available transportation systems. Past research in this area has been primarily upon "connective" models such as transportation models connecting supply with consumption and planning factor models connecting operations with material requirements. Although a number of studies of the multi-activity aspect has been made, no satisfactory integrated analytic investigation has been achieved so far as is known. An attempt by Clark (1963 [15]) in this direction did not proceed beyond a formulation of a single-activity problem.

4. Evaluation by Servomechanism Theory

Several studies have used portions of servomechanism theory to analyze the single-activity inventory control problem in cost-free systems. These studies have considered both deterministic and random demand. Early work assumed continuous (over time) inputs and outputs of the inventory system and used Laplace transforms as the analytic tool. Later work recognized the discrete nature of inventory processes and used z-transform methods.

The multi-activity inventory problem using the discrete time (sampled-data) servomechanism approach was investigated by Burns (1970 [11]). Although potentially the theory could be used for

inventory control purposes in the same sense as the other techniques described previously, the main thrust of Burn's study was to apply the method to evaluate the consequences of using a particular decision (ordering) rule at the various activities in a cost-free inventory system. The decision rule studied was that the desired level of inventory is expressed as a certain number of weeks worth of expected demand, the expectation being a first-order exponentially-smoothed average of past demands. For this decision rule, a model is constructed for the case of a series activity structure which is later interpreted in context with an arborescence structure. The model is used to show that the system amplifies minor variations in consumer demand into major disturbances at higher activities. This amplification is traced to two sources, one being a legitimate and unavoidable inventory adjustment and the other being an unwarranted "false order" effect. A new decision rule is then derived which automatically suppresses the false orders. The two decision rules are compared by use of a simulation model of a three-echelon system. It is demonstrated that the revised rule experiences fewer stockouts, maintains lower average inventories, and greatly reduces the amplification effect.

5. Evaluation by Simulation

All analytic inventory control models suffer from constraining assumptions concerning the environment within which the inventory system operates. Although simulation models can incorporate richer representations of the environment, they cannot solve inventory problems in the same sense as the analytic approaches. They are useful, however, in an evaluative or comparative role: the behavior of analytically derived policies can be studied, or alternative inventory policies can be compared in context with richer representations of the operating environment than were used in the derivation of the policies.

For multi-activity inventory problems, several simulation models (such as in Refs. [11] and [14]) have been constructed to provide realizations of a particular policy in essentially the same operating environment as that assumed in the derivation of the policy; the purpose here is to demonstrate the effects and behavior of the analytically derived policy. Other simulation models have been constructed for the explicit purpose of comparing alternative policies; a recent example is given by Haber (1971 [27]). In this multi-product simulation model, the operation of a particular three-level arborescence activity system (the Polaris submarine support system) is studied for four principle inventory policies. None of the policies tested were multi-activity policies in the sense of being derived from analytic multi-activity inventory theory; instead, they reflected variations of policies used in practice which tend to be aggregates of single-activity policies. The operation of the simulation model yields expected costs stratified by type of cost and type of policy, and various inventory performance statistics for each type of policy. From these results for the particular inventory system studied, judgements were made concerning the behavior and relative effectiveness of the alternative policies for various classes of products.

VI. DIRECTION OF FUTURE RESEARCH

It is probable that research in multi-activity inventory theory has reached a point where highest returns have already been achieved (the easy problems have been solved) and, therefore, marginal returns from further work are likely to diminish. The principle opportunities for further work, both by individual researchers and research teams, probably lie in refinments and extensions of previous results and in the reduction of currently available theory to practice in actual inventory situations. In addition, there may still be a small probability that a new basic theory can be developed which would supercede much of the previous results.

Before considering directions that future research in multi-activity inventory theory might take, some impressions of the current state of the art are given. In particular, an attempt is made to identify weaknesses in the current theory which future research might be able to overcome.

1. State of the Art

The review of research on the multi-activity inventory problem in previous sections has included studies which might be considered as obsolete in view of later progress. Although these studies were included to provide some kind of historic perspective, it is possible that some of the techniques used may play a role in future research. Furthermore, the earlier work tends to be simpler and therefore may be more viable with respect to practical use than the more sophisticated results of later research.

Looking at recent work on the deterministic inventory control problem, an impression is gained that the problem as formulated has been solved, at least from a theoretical point of view. The models and solution methods developed by Zangwill, Veinott, and others appear to encompass most of the important features of the problem. For a general problem of practical size, the amount of computation involved in finding solutions is much too great to be practical, but this is probably due more to the inherent complexity of the problem than to the theory. However, for special cases and assumptions that are not too unrealistic, efficient solutions methods have been developed which make the computational effort more tractable.

In looking at the various techniques that have been applied to the stochastic multi-activity problem, it is clear that the Bessler-Veinott approach is considerably more general than the others in terms of included features of the problem. The most important feature not included is economy of scale resulting from fixed costs, but this factor for actual inventory problems will probably decline in relevancy as management and production automation continues to increase. Another significant limitation is that conditions necessary for the myopic policies of this approach to be optimal are quite restrictive with respect to consideration of dynamic factors. Within these limitations, the main difficulty with the model is a computational one, again largely attributable to complexities resulting if the full power of the model is exercised. Limitations imposed to make the computation possible tend to be over-restrictive relative to practical situations. However, the theory is very useful in analyzing the characteristics of optimal policies for the multi-activity inventory control problem, particularly where stationary or near stationary conditions pertain.

A sufficient amount of experience with the dynamic programing approach to the multi-activity inventory control problem is now available as to confirm its computational practicability (see Refs. [10] and [45] for typical citations of computation experience). Unfortunately, this technique suffers from several inadequacies with respect to included features of the problem. The case of parallel activities is still bothersome, even though there is reason to believe good approximations are available. But the main trouble is that, so far, the method has been confined to single-product problems. Extensions to even simple multi-product situations have not yet been made.

The dynamic programing approach and, to a lesser degree, the Bessler-Veinott model consider nonstationary aspects which are very important in practical situations, but they both suffer from assumptions of periodic review. Everything else being the same, continuous review policies should result in less inventory costs than periodic review policies. The expected cost and stationary process approaches yield continuous review policies, but they cannot adequately cope with nonstationary factors. Although continuous review models have been developed for multi-product problems, only

simple product interdependencies (such as budget or capacity constraints) have been considered in contrast to the more general inter-product relationships encompassed by the Bessler-Veinott method. The main advantage of expected cost and stationary process models is that the computations involved in obtaining solutions are considerably less than for the dynamic technique.

If a survey were to be made of techniques currently used to control actual multi-activity inventory systems, the use of heuristic methods which, at best, are only slightly tempered by results of inventory theory will probably be found to prevail in the majority of the systems. For the remainder, expected cost models are probably used, but mostly in the form of combinations of single-activity models rather than an integrated system model. A claim is made Ref. [19] for the Sherbrooke model that it is one of the first analytic multi-activity models to be translated from theory to practice. Looking into the near term future, it is plausible that both the stationary and dynamic programing multi-activity techniques are candidates for practical implementation, but most probably with nonoptimal modifications to cope with realisms not encompassed by the theory. For the more general Bessler-Veinott approach, a wider base of familiarity with the method and further work to reduce the computation requirements is probably indicated prior to any serious implementation consideration.

Considering next the class of allocation models reviewed in section IV, it appears that the books are more or less closed with respect to this as a separate problem. Instead, this appears to be subsumed by the class of supply policies derived from the more general inventory control techniques. This is particularly evident in the Bessler-Veinott method, where supply policies are explicitly considered in a formal manner. On the other hand, it is easy to envision special allocation or redistribution problems for which the use of simpler methods is attractive.

The stock positioning, repair level, and contingency support problems discussed in section V all have a common denominator; namely, the network of activities in a multi-activity system is itself a decision variable as well as imbedded inventory control policies. Thus, the three problems cited may be viewed as special cases of a general problem that has an additional dimension of complexity over the multi-product, multi-activity inventory control problem (an appropriate descriptor for the additional dimension might be "multi-network"). It is probably fair to assert that the state of the art with respect to this larger problem is still primitive, as testified by the confinement of progress to date to special cases. Although the Williams model on the repair-level problem recognizes product interdependencies in terms of parts hierarchy relationships, the model assumes stationarity and inventory policies are not included in the form of decision variables. The Pincus model includes inventory policies in the form of sub-optimizations, permits only simple interproduct relationships, and does not consider regeneration through repair. As previously noted, analytic work on the general contingency problem, which is actually a dynamic version of the stock positioning problem, is virtually nonexistent.

2. Short-Term Research

In general, short-term opportunities for research on multi-activity inventory problems lie in extensions and refinements of previously developed approaches, and in several aspects of the problem of translating available theoretical results to practice. Of the several approaches to the inventory control problem, the ones that assume nonstationary stochastic conditions appear to be the most interesting.

In the Clark-Scarf approach, further investigations of the parallel activity case (and the general arborescence structure) might prove fruitful; in particular, the work of Zacks provides a good lead on this problem. Additional research here could take the form of (a) finding conditions such that relatively

simple policies are optimal, (b) determining sharper bounds on the optimal solution, or (c) developing good approximations to the optimal solutions. Another useful extension of this approach would be to permit simple inter-product dependencies such as budget or capacity constraints; it may also be possible to consider more detailed interrelationships by appropriate interpretations of activities as products in some manner analogous to the Bessler-Veinott development.

The Bessler-Veinott model is of special interest with respect to further research because it enables the inclusion of more factors of the problem than the other methods; its ability to cope with multi-product as well as multi-activity systems is particularly attractive. Even though several methods for alleviating computational difficulties of the model have been developed, including bounding, approximation, and special assumption methods, further research along these lines may be beneficial. In general, these observations apply also to the deterministic models of Zangwill and Veinott.

Since the use of network theory for the multi-activity inventory problem has only recently been opened up by Connors-Zangwill, it is difficult to judge what directions future research along this line might take. It may be that significant new developments will depend upon advances in general network theory. On the other hand, extensions and refinements within the framework already established may be possible.

It has been pointed out that continuous review assumptions have an advantage over periodic review. The general continuous review, dynamic problem is very difficult. However, further research along the lines of step-function approximations to continuous review policies may be useful. The rather heuristic method used in the Clark-Metcalfe model (referred to in section III–3) may provide a start in this direction. In this model, it is assumed that critical levels are computed periodically, but under the assumption that they are implemented in a continuous review mode throughout the period; in effect, this eliminates the implicit inflation of resupply time induced by the periodicity assumption. A study of this and other approximations to the optimal continuous review policies might provide usable results.

As previously indicated, further research on allocation and distribution problems is most likely to be accomplished in context with the more general inventory control problem. For the repair level and other special problems defined in section V, further work will undoubtedly be done as extensions of the previous studies, but research on the general problem of which the cited problems are special cases is of a more long-term nature. Insofar as evaluation techniques are concerned, simulation is always available for use in a variety of evaluation and planning problems, but it is hard to categorize such applications per se as theoretical research. Although some additional work on the use of servomechanism theory may occur for evaluation or inventory control purposes, it will probably be in context with special kinds of products or systems.

Even though it may not be appropriate to view efforts to reduce multi-activity inventory theory to practice as theoretical research, this field should not be overlooked. The adaptation of the theory in the form of more-or-less heuristic models of multi-activity systems that are intuitively appealing for practical application provides important opportunities for further work. Also, the susceptibility of a particular analytic method with respect to its practical use may be considered as some kind of ultimate criterion of its worth.

3. Long-Term Research

Over the long run, the real frontier for research on multi-activity inventory systems lies in the area of integrated logistics, where the surface so far has barely been scratched. In many organizations,

the pure inventory problem does not exist and an inventory model, no matter how comprehensive in terms of activity structures, cost functions, and optimization, fall considerably short of satisfying overall logistics objectives. A full appreciation of the interplay among operations, maintenance, transportation, and supply, all being tempered and affected by associated management and data processing systems, tends to relegate pure inventory models to a less relevant status. Even though the full problem is exceedingly complex, there are several avenues worthwhile exploring by analytic techniques.

As a start in this direction, the extension of existing theory to what might be called the multi-network problem (which, by definition, includes multi-product and multi-activity considerations) could be contemplated. This problem is roughly defined as one where parts hierarchy structures and related reliability and end-item useage factors are given, together with a network representation of the potential logistics support system. It is then desired to find a substructure of the parts hierarchy for which inventories are to be provided and, for each such item, a subnetwork of the support system at which inventories are to be maintained such that total system costs are minimized. Theoretically, this problem encompasses almost all of the multi-activity inventory problems so far investigated. Also, to solve this problem properly, the notion of "inventory" should be generalized to include such commodities as manpower, maintenance equipment, and other support resources.

In dealing with this problem, it is first necessary to define it in more rigorous terms. Next, an appropriate solution technique must be found. Of the ones now available, the Bessler-Veinott model may be a possible point of departure since it enables the potential consideration of dependencies among items in a parts hierarchy. Unfortunately, however, the problem is characterized by features involving economies of scale which this technique cannot handle; some way around this difficulty would have to be found.

Assuming that a formal model for the problem can be constructed, the next step would be to look for possible decomposition opportunities which would almost certainly be necessary if any use were to be made of the model. Also, if this stage is reached, there would probably be a number of research possibilities with respect to special cases and computation procedures, sensitivity analyses, etc. Even though the full multi-network problem may turn out to be insolvable in any practical sense, a formal recognition of the problem is useful in identifying just what kinds of suboptimizations are being made in current and future multi-activity inventory studies.

Although a solution to the multi-network problem would be a big step toward the full integrated logistics problem, it would still contain rather abstract representations of the real-world processes. It is quite possible that advantages of analytic optimization techniques would be overcome by effects of features not included or inadequately represented in the formulation; whether or not this is true would be very difficult to prove. As a result, it might be profitable to resort to a decision-making apparatus which might be called "micro-simulation."

The use of the micro-simulation technique is based upon a recognition of an advanced real-time management system as being, in effect, a "model" of the real-world system being managed. Since such a management system contains status records in varying degrees of detail, supported by data flows to keep the records current, it is theoretically possible to add components that would simulate the external stresses upon the system as caused by the real-world operations. With such additions, it would then be possible to make short-term accelerated-time, simulated "excursions" into possible futures of the system for decision-making purposes. Such micro-simulations might employ evaluation-selection procedures for postulated alternatives, or they might contain adaptive decision-making models.

A very crude example of the latter possibility is given by a model developed by Fisher, et al. (1968 [23]). This model, which is primarily a Monte Carlo simulation model for military logistics applications, contains two innovative features: the use of task networks to describe operational and support environments, and the use of an embedded decision model for resources determinations.

The task network concept involves the establishment, by user-provided data, of networks of interconnected tasks or activities to represent equipment utilization, repair and overhaul functions, and other aspects of the operational and logistics environment. The user also identifies types and amounts of resources needed to accomplish each kind of task; categories of included support resources are maintenance personnel, repair equipment, and spare parts. Other input data includes time distributions for task accomplishment, priority rules, shift policies, end-item utilization schedules, etc. With this data, the model simulates the operation of end-equipment, the incurrence of malfunctions, the accomplishment of servicing tasks, the accomplishment of repair and overhaul functions, the utilization and interaction of resources in the demand process, the changes in resource availability according to shift policies, and other facets of the overall operation. In the simulation, inventories of spare parts are treated identically the same as inventories of other needed resources such as maintenance specialists, shop equipment, etc. Also, the model permits tradeoffs within and among the various types of resources whenever shortages occur; included is task preemption, use of substitutes, task deferment, cannibilization, etc.

The embedded decision model was designed to operate during the course of a particular simulation according to criteria established at the outset. Its general function is to identify particular resource items (bottleneck items) tending to inhibit the achievement of overall end-item utilization objectives and to cause selective injections of additional amounts of these resources according to a cost-effectiveness criterion. Thus, the "reordering" of a resource is a consequence not of before-the-fact demand estimates, but rather of need against end objectives after all reasonable advantages are taken of resource tradeoffs, deferment of needs, etc. The decision model uses an automatic exponential-smoothing forecasting technique to predict degradation in system performance, and a simple marginal analysis procedure for resource augmentations.

Extrapolating the methodology of this model to the micro-simulation procedure, the simulation aspects would be replaced by the real-time management system, augmented by components for simulating external stresses. It is clear, however, that this procedure, even though extensively refined and supported by analytic models and micro-simulations for decision-making over long-term futures of the system, cannot achieve "optimal" policies or resource allocation decisions in the same sense as in purely analytic techniques. Nevertheless, it may represent a viable approach with respect to the practical aspects of the integrated logistics problem.

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ON A SEQUENTIAL TEST PROCEDURE WITH DELAYED OBSERVATIONS*

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ABSTRACT

This paper considers sequential test procedures to decision problems where there exists time delays in obtaining observations.

1. INTRODUCTION

The investigation of strategies relating to sequential decision problems has become of increasing importance. It is widely recognized that many decision situations require action sequentially in time. A closely related set of questions is concerned with stopping rule problems. In this case, it must be decided whether a program of observation, or action, should be continued to obtain further information or be terminated and some final action taken. Stochastic sequential decision situations are different from the usual decision problems insofar as the decision maker can look at a sequence of observation a group at a time and to decide after each observation group whether to stop sampling and to take action immediately or to continue sampling and postpone action until a later time. In general, he must compare the loss in stopping now with the expected loss were he to continue. These situations have received much attention recently, in connection with stopping rule problems.

A variety of basic assumptions are usually made in sequential decision models. These are often a tremendous limitation on the applicability of the theory.

One basic assumption often made in stochastic sequential decision problems, is that the results of taking observations are considered as immediately available. In other words, it is assumed that there is no time delay between the decision to take an observation and obtaining the result of the observations. In actuality, such time lags can be substantial. In practice, taking an observation often involves experimention. One example, where this time delay often considerably inhibits the use of sequential analysis, relates to medical experimentation. Here, a long time may elapse between the application and the result of a treatment. Another important area, having similar problems with time delay, is that of industrial experimentation.

This paper studies a situation where observations, with random time delay, are taken at the beginning of the experiment and a decision is to be made before a specified time T^*. In section 2, a description of the model and related assumptions is given. In section 3, an optimal stopping rule is obtained and it is shown that the conditional minimum expected cost function satisfied a recursive functional equation. Finally, in section 4 the optimal initial sample size is discussed and a computational method and some properties of the optimal stopping rule is provided.

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2. MODEL FORMULATION AND PRELIMINARIES

We are given a statistical decision problem (Ω, A, L), with a prior distribution, c_o. At time $t=0$ a random sample of size N_o is taken, where Ω is the parameter space, A the action space and L the loss function defined for pairs (Θ, a) where $\Theta\epsilon\Omega$ and $a\epsilon A$.

The observations $X_1\ X_2,\ .\ .\ .,\ X_{N_0}$ are independently and identically distributed with common distribution function $P_\Theta(x)$, where Θ is an unknown parameter in Ω. Associated with each observation, X_i, there is a random time delay, Y_i. The X_i's and Y_i's are mutually independent, and Y_i's are independently and identically distributed with a common known distribution function, $H(y)$. We assume that the density function, $h(y)$, exists and is continuous for $y\geq 0$. Because of the time delay, we are, now, unable to obtain the outcomes of all observations immediately. If there is no cost associated with time, the optimal test procedure is to postpone terminal decision until all outcomes are obtained. However, if there is a cost associated with time, then at any point of time, we are faced with a problem of whether to continue or to stop the experiment. We will gain more information and possibly reduce the expected cost due to wrong terminal decision by continuing the experiment; while, on the other hand, we incur additional cost on time. Hence, it becomes a problem of balancing the quality of the terminal decision and the cost of experimental time. In this paper, we assume a linear cost of b for time. We also assume there exists a conjugate family of distributions, C, defined on the parameter space, Ω, such that it is closed under sampling, i.e., the prior distribution and any possible posterior distributions resulted are members of this family. For each $c\epsilon C$, we assume that the Bayes decision with respect to c exists, and the Bayes risk, $B(c)$, is bounded. We assume there exists a time limit, T^*, on the experiment. In other words, we assume that the experiment cannot be carried beyond the time, T^*. Finally, to make this limit meaningful, we assume $H(T^*)<1$.

With all these assumptions, a test procedure should consist of a rule of the initial sample size N_o, a stopping rule and a terminal decision rule.

It is known for the non-time-delay sequential case, that the optimal terminal decision rule is independent of the stopping rule used. In other words, it is proved that Bayes decision should always be the terminal decision once the experiment is stopped. It is easy to extend the argument to the time-delay case, since the expected cost due to wrong terminal decision is not affected at all by the assumption of cost on time. Hence, evidently, Bayes rule will be the optimal terminal decision rule. Once the experiment is stopped, we should take Bayes decision with respect to c as the terminal decision, where c is the then posterior distribution, and take a loss of $B(c)$, the Bayes risk.

At any time, t, during the experiment, the relevant information is summarized by the current posterior distribution, c_t, and the number of observations currently outstanding, N_t. An observation is regarded as outstanding if its outcome has not been obtained yet. With c_t we can calculate the Bayes risk $B(c_t)$, if we stop. With c_t and N_t we can calculate the additional expected cost if the experiment is continued.

A stopping rule, designated by $\delta(t, n, c)$, is defined as a two valued function, 0 and 1, where $0\leq t\leq T^*$, $n=0,\ 1,\ 2,\ .\ .\ .,\ N_o$ and $c\epsilon C$. At any point in time, t, we continue the experiment as long as $\delta(t, N_t, c_t)=1$ and stop as soon as its value is 0.

Let $[t, n, c]$ denote the condition that the experiment is not stopped before t and at t, the number of observations outstanding is n, $N_t=n$, and the posterior distribution is c, $c_t=c$. Also, for $t>0$ let $\varphi_\delta(t, n, c)$ denote the expected cost, excluding the cost incurred before t, given that the stopping rule, δ, is used and the condition, $[t, n, c]$. For $t=0$, let $\varphi_\delta(o, n, c)$ denote the expected cost excluding the cost of

observations given that the stopping rule δ is used, $N_o=n$ and $c_o=c$. Hence, given a stopping rule, δ, the total expected cost, $\mathscr{L}(\delta)$, is

$$\mathscr{L}(\delta)=\varphi_\delta(o, N_o, c_o)+a\cdot N_o, \tag{2.1}$$

where a is the unit cost for observation. We want to find a stopping rule δ_o, such that

$$\mathscr{L}(\delta_o)=\varphi_{\delta_o}(o, N_o, c_o)+a\cdot N_o \leqslant \mathscr{L}(\delta)=\varphi_\delta(o, N_o, c_o)+a\cdot N_o \tag{2.2}$$

for all $\delta \in \delta(T^*)$, where $\delta(T^*)$ is the set of all stopping rules which stop unconditionally at T^*, that is,

$$\delta(T^*)=\{\delta \mid \delta(T^*, n, c)=0, \quad \text{for all } n \text{ and } c\in C\}. \tag{2.3}$$

The conditional minimum expected cost function is defined by,

$$\varphi(t, n, c)=\inf_{\delta\in\delta(T^*)} \varphi_\delta(t, n, c). \tag{2.4}$$

Before we go into details, we introduce a few more notations. We will use $h_t(y)$ to denote the conditional density function of time delay after t given that it is greater than t, i.e.,

$$h_t(y)=h(t+y)/1-H(t). \tag{2.5}$$

$h_{i,t}(y)$ will be used to denoted the density function of the waiting time spent ater t for obtaining the first outcome after that, given that at t, $N_t=i$. That is

$$h_{i,t}(y)=\frac{d}{dy}H_{i,t}(y)=\frac{d}{dy}\left[1-\left(1-\frac{H(t+y)}{1-H(t)}\right)^i\right] \tag{2.6}$$

$$=i\cdot(1-H_t(y))^{i-1}h_t(y).$$

Given a prior distribution, c, let $c(x)$ denote the posterior distribution obtained by observing an outcome, x.

3. OPTIMAL STOPPING RULE

Let us consider initially what to do when there are no observations outstanding. It is clear that we should stop when the last outcome is obtained since, in no event, will the Bayes risk decrease by continuing. Hence,

$$\varphi(t, o, c)=B(c) \tag{3.1}$$

for all $0\leqslant t\leqslant T^*$ and $c\in C$. When there are observations outstanding, the solution is not that straightforward.

Suppose now that we know how to behave optimally for all $0 \leq t \leq T^*$ and $c \in C$ for the case when $N_t \leq n$. What should we do at $[t, n+1, c]$? Consider the following rule where a stopping point $z \geq t$ is chosen and if no results arrive in (t, z) then stop. However, if one of the $n+1$ outstanding results arrives before then, leaving only n outstanding results, behave optimally. In this way we arrive at $[t', n, c(x)]$, where $t < t' < z$ and we know how to behave optimally. The expected cost, $\varphi_z(t, n+1, c)$ is obtained by means of conditional expectations and is given as

(3.2)

$$\varphi_z(t, n+1, c) = \int_0^{z-t} \{E_x[\varphi(t+y, n, c(x))] + b \cdot y\} h_{n+1,t}(y)\, dy + [1 - H_{n+1,t}(z-t)]\,[B(c) + b(z-t)],$$

where functions $h_{n+1,t}$ and $H_{n+1,t}$ are defined in (2.6), φ is defined in (2.4), and b is the unit cost for time. The expected value operator is with respect to $P_c(x)$ the distribution of X given that the posterior distribution of Θ is c. It will later be shown that

$$\varphi(t, n+1, c) = \operatorname*{Inf}_{t \leq z \leq T^*} \varphi_z(t, n+1, c); \tag{3.3}$$

this idea suggests a recursive method for obtaining the optimal stopping procedure. With this in mind we introduce a function $g(t, n, c)$ with $g(t, o, c) = B(c)$ and satisfying the following recursive relation:

$$g(t, n, c) = \operatorname*{Inf}_{t \leq z \leq T^*} \left\{ \int_0^{z-t} [E_x(g(t+y, n-1, c(x)) + b \cdot y] h_{n,t}(y)\, dy + [1 - H_{n,t}(z-t)] \cdot [B(c) + b(z-t)] \right\}. \tag{3.4}$$

We note that the function $g(t, n, c)$ is completely defined once we specify $g(t, o, c) = B(c)$.

The basic concern in the remainder of this section is to show that $g(t, n, c) = \varphi(t, n, c)$ and that the optimal stopping rule can defined in terms of $g(t, n, c)$ as follows:

$$\delta(t, n, c) = \begin{cases} 1 & \text{if } g(t, n, c) < B(c) \\ 0 & \text{if } g(t, n, c) = B(c). \end{cases} \tag{3.5}$$

That is, we stop at $[t, n, c]$ as soon as $g(t, n, c) = B(c)$.

DEFINITION: Given a stopping rule, δ, $t_\delta^*(t, n, c)$ is called the first stopping point at (t, n, c), if

$$t_\delta^*(t, n, c) = \sup \{\{t\} \cup \{t_0 | t_0 \geq t, \delta(t_0, n, c) = 1 \text{ and } \delta(t_1, n, c) = 1, \quad \text{for all } t \leq t_1 < t_0\}\}, \tag{3.6}$$

where $\{t\}$ is the set that contains a single point, t.

$t_\delta^*(t, n, c)$ will depend on δ, t, n, and c. However, for simplicity, we will use t_δ^* to represent $t_\delta^*(t, n, c)$ if there is no danger of confusion. Physically, $t_\delta^*(t, n, c) = t$ if the experiment is stopped at the point (t, n, c) under δ. If the experiment is continued, then we are waiting for the next outcome. As time elapses, if the next outcome does not arrive, sooner or later we will reach a point where the

stopping rule, δ, tells us to stop. In this case, t_δ^* will be that point. t_δ^* will be used later in calculating conditional expected cost.

LEMMA 3.1: Given any stopping rule, $\delta \in \delta(T^*)$, $g(t, n, c) \leq \varphi_\delta(t, n, c)$ and $g(t, n, c) \leq \varphi(t, n, c)$ for all (t, n, c).

PROOF: We will use mathematical induction to prove the lemma. Given any δ, when $n = 0$, from (2.4) and (3.1) $\varphi_\delta(t, o, c) \geq \varphi(t, o, c) = B(c) = g(t, o, c)$. Hence the lemma is true when $n = o$. Now, suppose it is true for $n = i$. We will show that it is also true for $n = i + 1$. From conditional expectation,

$$\begin{aligned}
\varphi_\delta(t, i+1, c) &= \int_o^{t_\delta^* - t} \{E_x[\varphi_\delta(t+y, i, c(x))] + b \cdot y\} h_{i+1,t}(y)dy \\
&\quad + [1 - H_{i+1,t}(t_\delta^* - t)] \cdot [B(c) + b(t_\delta^* - t)] \\
&\geq \int_o^{t_\delta^* - t} \{E_x[g(t+y, i, c(x))] + b \cdot y\} h_{i+1,t}(y)dy \\
&\quad + [1 - H_{i+1,t}(t_\delta^* - t)][B(c) + b(t_\delta^* - t)] \\
&\geq g(t, i+1, c).
\end{aligned}$$

The last inequality is obtained from the definition of $g(t, i + 1, c)$. Hence, the lemma is true for all (t, n, c). From the preceding it immediately follows that $g(t, n, c) \leq \varphi(t, n, c)$.

We now derive some properties of $g(t, n, c)$. For each (t, n, c) let us define a function of z

(3.7)
$$G_n(t, c, z) = \int_o^{z-t} \{E_x[g(t+y, n, c(x)] + b \cdot y\} h_{n+1,t}(y)dy + [1 - H_{n+1,t}(z-t)][B(c) + b(z-t)],$$

where $t \leq z \leq T^*$. With this notation

(3.8)
$$g(t, n, c) = \inf_{t \leq z \leq T^*} G_{n-1}(t, c, z).$$

LEMMA 3.2: $g(t, n, c)$ is a bounded function.

PROOF: From (3.8), for $n \geq 1$, $g(t, n, c) = \inf_{t \leq z \leq T^*} G_{n-1}(t, c, z) \leq G_{n-1}(t, c, t) = B(c)$. When $n = 0$, $g(t, o, c) = B(c)$. Since we assumed that Bayes risk is bounded, $g(t, n, c)$ must be bounded.

LEMMA 3.3: Given a fixed n if for every $c \in C$, $g(t, n, c)$ is a continuous function of t; then given any $\epsilon_1 > 0$ there exist a $\epsilon_2 > 0$ such that, within the domain of definition, the following three statements are true:

(1) $|G_n(t, c, z_1) - G_n(t, c, z_o)| < \frac{\epsilon_1}{2}$, whenever $|z_1 - z_o| < \epsilon_2$;

(2) $|G_n(t + \Delta t, c, z_o + \Delta t) - G_n(t, c, z_o)| < \frac{\epsilon_1}{2}$, whenever $|\Delta t| < \epsilon_2$ and $t \leq z_o \leq T^*$;

(3) $|G_n(t - \Delta t, c, T^* - t_o) - G_n(t - \Delta t, c, T^* - \Delta t)| < \frac{\epsilon_1}{2}$, whenever $0 \leq t_o \leq \Delta t < \epsilon_2$.

PROOF: From (3.7)

$$G_n(t, c, z)=\int_o^{z-t}\{E_x[g(t+y, n, c(x))]+b\cdot y\}h_{n+1,t}(y)\,dy+[1-H_{n+1,t}(z-t)]\cdot[B(c)+b(z-t)].$$

Since, $g(t, n, c)$ is a continuous function of t for all $c\in C$ $g(t, n, c)$ is bounded and $h_{n+1,t}(y)$ is continuous by assumption, it is evident that for each fixed (t, n, c), $G_n(t, c, z)$ is a continuous function of z over the closed interval, $[t, T^*]$. Hence it is uniformly continuous over that closed interval. This proves statement 1. We will not give the proof for statement 2 and 3, however, they can be done by using the continuity and boundedness property and going through limiting process.

During the proof, we have shown that $G_n(t, c, z)$ is a continuous function of z, if $g(t, n, c)$ is a continuous function of t for every $c\in C$. This implies, in that case

$$g(t, n+1, c)=\inf_{t\le z\le T^*} G_n(t, c, z)=\min_{t\le z\le T^*} G_n(t, c, z). \tag{3.9}$$

THEOREM 3.1: Given any fixed n and c, $g(t, n, c)$ is a continuous function of t.

The proof of this result is given in the appendix.

LEMMA 3.4: For every (t, n, c)

$$g(t, n+1, c)=\min_{t\le z\le T^*} G_n(t, c, z), \tag{3.10}$$

and $G_n(t, c, z)$ is a continuous function of z, for $t\le z\le T^*$.

PROOF: It is the result of Theorem 3.1.

In order to obtain an optimal stopping rule, consider the following rule,

$$\delta(t, n, c)=\begin{cases}0 & \text{if } g(t, n, c)=B(c)\\ 1 & \text{otherwise.}\end{cases} \tag{3.11}$$

If we can show that $\varphi_\delta(t, n, c)=g(t, n, c)$, then we can conclude that δ is an optimal stopping rule, because we have shown $g(t, n, c)\le\varphi(t, n, c)$.

THEOREM 3.2: $\varphi_\delta(t, n, c)=g(t, n, c)$ for all (t, n, c), where δ is the stopping rule defined in (3.11).

PROOF: We will use mathematical induction to prove the theorem. When $n=0$, $g(t, o, c)=B(c)$. This means, from (3.11), $\delta(t, o, c)=0$, i.e., a decision to stop the experiment. Hence, $\varphi_\delta(t, o, c)=B(c)$. This implies that the theorem is true for $n=0$. Now, suppose the theorem is true for $n=i$, i.e., suppose $g(t, i, c)=\varphi_\delta(t, i, c)$, for all t and $c\in C$. We want to show that it will also be true for $n=i+1$. When $g(t, i+1, c)=B(c)$, from (3.11), $\delta(t, i+1, c)=0$. Hence, in this case, $\varphi_\delta(t, i+1, c)=B(c)=g(t, i+1, c)$. Now, suppose for some t and c, $\varphi_\delta(t, i+1, c)\neq g(t, i+1, c)$. It is obvious that $g(t, i+1, c)\neq B(c)$ and $\delta(t, i+1, c)=1$. This means there exists a z_o, $t<z_o<T^*$, such that

$$g(t, i+1, c)=\min_{t\le z\le T^*} G_i(t, c, z)=G_i(t, c, z_0). \tag{3.12}$$

Using conditional expectation,

$$\varphi_\delta(t, i+1, c) = \int_0^{t_\delta^*-t} \{E_x[\varphi_\delta(t+y, i, c(x))] + b \cdot y\} h_{i+1,t}(y)dy \tag{3.13}$$

$$+ [1-H_{i+1,t}(t_\delta^*-t)][B(c)+b(t_\delta^*-t)]$$

$$= \int_0^{t_\delta^*-t} \{E_x[g(t+y, i, c(x))] + b \cdot y\} h_{i+1,t}(y)dy$$

$$+ [1-H_{i+1,t}(t_\delta^*-t)][B(c)+b(t_\delta^*-t)]$$

$$= G_i(t, c, t_\delta^*) \geq g(t, i+1, c) = G_i(t, c, z_0),$$

where t_δ^* is the first stopping point at $(t, i+1, c)$. Since $g(t, i+1, c) \neq \varphi_\delta(t, i+1, c)$, (3.13) implies $G_i(t, c, t_\delta^*) > G_i(t, c, z_0)$. Hence, $t_\delta^* \neq z_0$

CASE 1: $z_0 < t_\delta^*$.

For any z, $z_0 < z \leq T^*$, it can be shown that

$$G_i(t, c, z) = G_i(t, c, z_0) + [1-H_{i+1,t}(z_0-t)][G_i(z_0, c, z) - B(c)]. \tag{3.14}$$

This together with (3.12) imply $[1-H_{i+1,t}(z_0-t)][G_i(z_0, c, z)-B(c)] \geq 0$. Since we assumed $H(T^*) < 1$, this implies $G_i(z_0, c, z) \geq B(c)$, for any z, $z_0 < z \leq T^*$. Hence, $g(z_0, i+1, c) = \underset{z_0 \leq z \leq T^*}{\text{Min}} G_i(z_0, c, z) = G_i(z_0, c, z_0) = B(c)$. In other words, $\delta(z_0, i+1, c) = 0$. However, since $t < z_0 < t_\delta^*$, this contradicts the definition of t_δ^*. Hence, $z_0 < t_\delta^*$ cannot be true.

CASE 2: $z_0 > t_\delta^*$.

Since $G_i(t, c, z)$ is a continuous function of z, we can find a $\Delta t > 0$ such that for all t_0, $t_\delta^* \leq t_0 < t_\delta^* + \Delta t < z_0$.

$$G_i(t, c, t_0) > G_i(t, c, z_0). \tag{3.15}$$

However, $G_i(t, c, z_0) = G_i(t, c, t_0) + [1-H_{i+1,t}(t_0-t)] \cdot [G_i(t_0, c, z_0) - B(c)]$. From (3.15), this implies $G_i(t_0, c, z_0) < B(c)$, for all t_0, $t_\delta^* \leq t_0 < t_\delta^* + \Delta t$. This means $g(t_0, i+1, c) < B(c)$ and $\delta(t_0, i+1, c) = 1$, for all $t_\delta^* \leq t_0 < t_\delta^* + \Delta t$. This is a contradiction to the definition of t_δ^*. Hence $z_0 > t_\delta^*$ cannot be true.

From case 1 and 2, it is evident that no such t and c can exist with $\varphi_\delta(t, i+1, c) \neq g(t, i+1, c)$. From mathematical induction this proves the theorem.

LEMMA 3.5: $\varphi(t, n, c) = g(t, n, c)$ for all (t, n, c) and the stopping rule, δ, defined in 3.11 is an optimal stopping rule.

PROOF: In lemma 3.1, it was proven that $g(t, n, c) \leq \varphi(t, n, c)$. It has been shown in the preceeding theorem that $\varphi_\delta(t, n, c) = g(t, n, c)$. Hence, it is obvious that $\varphi(t, n, c) = g(t, n, c)$ and δ is an optimal stopping rule.

LEMMA 3.6: The conditional minimum expected cost function, $\varphi(t, n, c)$ satisfies the following recursive relation:

$$\varphi(t, n+1, c) = \underset{t \leq z \leq T^*}{\text{Min}} \left\{ \int_0^{z-t} \{E_x[\varphi(t+y, n, c(x))] + b \cdot y\} h_{n+1,t}(y)dy \right. \tag{3.16}$$

$$\left. + [1-H_{n+1,t}(z-t)][B(c)+b(z-t)] \right\}.$$

PROOF: This is obvious from the preceding lemma.

The optimal sample size N_o can now be obtained. We have derived a method for finding $\varphi(t, n, c)$. Now, given a statistical decision problem with prior distribution, c_o, the initial sample size should be chosen to minimize

$$\varphi(o, N_o, c_o) + aN_o. \tag{3.17}$$

It can be noted that N_o need never be chosen greater than $B(c_o)/a$

4. A COMPUTATION METHOD FOR THE OPTIMAL STOPPING RULE AND SOME CHARACTERISTICS

At a first glance of the recursive function, one might feel that it will be a tremendous job to calculate the conditional minimum expected cost function, $\varphi(t, n, c)$, because of the minimization involved. For that reason, a computation method has been developed. Although it is still an iterative method, it is much easier to compute. Let $\hat{\delta}$ denote the optimal stopping rule defined in (3.11).

LEMMA 4.1: Given any (t, n, c), $\hat{\delta}(t, n, c) = 1$, if there exists a t_o, $t < t_o \leq T^*$, such that $G_{n-1}(o, c, t) > G_{n-1}(o, c, t_o)$. If no such t_o exists, $\hat{\delta}(t, n, c) = 0$.

PROOF: Suppose there exists a t_o, $t < t_o \leq T^*$, and $G_{n-1}(o, c, t) > G_{n-1}(o, c, t_o)$. We know

$$G_{n-1}(o, c, t_o) = G_{n-1}(o, c, t) + [1 - H_{n,o}(t)] \cdot [G_{n-1}(t, c, t_o) - B(c)]. \tag{4.1}$$

This implies $G_{n-1}(t, c, t_o) < B(c)$, which again implies $g(t, n, c) < B(c)$. Hence, from (3.11), $\hat{\delta}(t, n, c) = 1$. Now, suppose there does not exist such a t_o. This means that for all t_1, $t < t_1 \leq T^*$, $G_{n-1}(o, c, t) \leq G_{n-1}(o, c, t_1)$. We know

$$G_{n-1}(o, c, t_1) = G_{n-1}(o, c, t) + [1 - H_{n,o}(t)] \cdot [G_{n-1}(t, c, t_1) - B(c)]. \tag{4.2}$$

This implies $G_{n-1}(t, c, t_1) \geq B(c)$, for all $t < t_1 \leq T^*$, which again implies $g(t, n, c) = B(c)$. Hence $\hat{\delta}(t, n, c) = 0$.

This is a really convenient result. Now, we don't have to go through that complex recursive function in order to obtain the conditional minimum expected cost function. First, we calculate $G_o(o, c, z)$, for all $c \in C$. This simple calculation will immediately identify the optimal stopping rule for the case where there is only one observation outstanding. With this stopping rule, we compute $\varphi(t, 1, c)$ which will be used to calculate $G_1(o, c, z)$. The computation involved here will be much easier than directly calculating $\varphi(t, 1, c)$ through the recursive function. We have thus avoided a minimization operation. Repeating the same procedure, we obtain the optimal stopping rule.

From lemma 4.1, we have seen that the optimal stopping rule depends on the behavior of $G_n(o, c, z)$ as a function of z. Hence it will be nice if the first partial derivative with respect to z exists, because it contains information about maximum and minimum points. From (3.7), we find the first partial derivative exists and

$$\frac{\partial}{\partial z} G_n(t, c, z) = [E_x[\varphi(z, n, c(x)) - B(c)] \cdot h_{n+1,t}(z - t) + [1 - H_{n+1,t}(z - t)] \cdot b. \tag{4.1}$$

(4.1) suggests that the failure rate, $h_{n+1,t}(z-t)/1-H_{n+1,t}(z-t)$, might play an important role. The answer is yes; however, because of the lengthy proof, only the result will be stated. Details can be found in [5].

LEMMA 4.2: If the failure rate, $h(y)/1-H(y)$, of the time delay distribution, $H(y)$, is nonincreasing, then, for each fixed n and $c\epsilon C$, $\varphi(t, n, c)$ is a nondecreasing function of t.

The above lemma can be put in a different form.

LEMMA 4.3: If the failure rate, $h(y)/1-H(y)$, of the time delay distribution, $H(y)$, is nonincreasing, then, for each fixed n and $c\epsilon C$, there exists a closed interval (the interval may be a single point), $[e(n, c), T^*]$, where $0 \leq e(n, c) \leq T^*$, such that $\hat{\delta}(t, n, c) = 0$ if and only if t is in $[e(n, c), T^*]$.

5. APPENDIX

THEOREM 3.1: Given any fixed n and c, $g(t, n, c)$ is a continuous function of t.

PROOF: We will use mathematical induction to prove it. When $n=0$, we have $g(t, 0, c) = B(c)$. This means that the theorem is true when $n=0$. Now, suppose that the theorem is true for $n=i$. This implies, from (3.9),

$$g(t, i+1, c) = \operatorname*{Min}_{t \leq z \leq T^*} G_i(t, c, z). \tag{5.1}$$

We want to show that the theorem will also be true for $n=i+1$. Let us assume that for some c_o and t_o, $g(t_o, i+1, c_o)$ is not continuous. This means that for some $\epsilon_1 > 0$ there exists a sequence, $\{\Delta t_i\}$, such that

$$\lim_{i \to \infty} \Delta t_i = 0 \text{ and } |g(t_o, i+1, c_o) - g(t_o + \Delta t_i, i+1, c_o)| > \epsilon_1. \tag{5.2}$$

We will show that such a sequence, $\{\Delta t_i\}$, can not exist. From the sequence, let us pick a subsequence, $\{\Delta t_{i_k}\}$, such that all terms in that subsequence will be either positive or negative.

CASE 1: All $\Delta t_{i_k} > 0$.

First, let us choose a $\epsilon_2 > 0$, such that all three statements in the lemma 3.3 will be true. Pick a Δt_j from the subsequence such that $\Delta t_j < \epsilon_2$ and $\Delta t_j < \frac{1}{2}(T^* - t_o)$. Such Δt_j will always exist, because $g(t, i+1, c)$ is defined for $0 \leq t \leq T^*$. If $t_o = T^*$, then no terms in the sequence can be positive. We know $|g(t_o, i+1, c_o) - g(t_o + \Delta t_j, i+1, c_o)| > \epsilon_1$. Yet, we do not know which term is larger. Both situations will be studied.

$$g(t_o, i+1, c_o) > g(t_o + \Delta t_j, i+1, c_o) + \epsilon_1: \tag{1.A}$$

Let z_o, $t_o + \Delta t_j \leq z_o \leq T^*$, be the point where

$$G_i(t_o + \Delta t_j, c_o, z_o) = \operatorname*{Min}_{t_o + \Delta t_j \leq z \leq T^*} G_i(t_o + \Delta t_j, c_o, z) = g(t_o + \Delta t_j, i+1, c_o). \tag{5.3}$$

Since $|\Delta t_j| < \epsilon_2$, from the statement 2 in lemma 3.3

$$|G_i(t_o + \Delta t_j, c_o, z_o) - G_i(t_o, c_o, z_o - \Delta t_j)| < \frac{\epsilon_1}{2}. \tag{5.4}$$

This together with (5.3) imply

$$(5.5)\qquad G_i(t_o, c_o, z_o-\Delta t_j) < G_i(t_o+\Delta t_j, c_o, z_o)+\frac{\epsilon_1}{2}=g(t_o+\Delta t_j, i+1, c_o)+\frac{\epsilon_1}{2}<g(t_o, i+1, c_o).$$

This contradicts that $g(t_o, i+1, c_o)=\underset{t_o\le z\le T^*}{\text{Inf}} G_i(t_o, c_o, z)$. Hence, (1.A) cannot be true.

$$(1.B)\qquad g(t_o+\Delta t_j, i+1, c_o) > g(t_o, i+1, c_o)+\epsilon_1:$$

Let z_o, $t_o \le z_o \le T^*$, be the point where

$$(5.6)\qquad G_i(t_o, c_o, z_o)=\underset{t_o\le z\le T^*}{\text{Min}} G_i(t_o, c_o, z)=g(t_o, i+1, c_o).$$

If $z_o+\Delta t_j \le T^*$, then from the statement 2 of lemma 3.3

$$(5.7)\qquad |G_i(t_o+\Delta t_j, c_o, z_o+\Delta t_j)-G_i(t_o, c_o, z_o)|<\frac{\epsilon_1}{2}.$$

This together with (5.6) imply $G_i(t_o+\Delta t_j, c_o, z_o+\Delta t_j) < G_i(t_o, c_o, z_o)+\frac{\epsilon_1}{2}=g(t_o, i+1, c_o)+\frac{\epsilon_1}{2}<g(t_o+\Delta t_j, i+1, c_o)$. This contradicts that $g(t_o+\Delta t_j, i+1, c_o)=\underset{t_o+\Delta t_j\le z\le T}{\text{Inf}} G_i(t_o+\Delta t_j, c_o, z)$. On the other hand, if $\Delta t_j+z_o>T^*$, since Δt_j was so chosen such that $\Delta t_j<\frac{1}{2}(T^*-t_o)$, we have $z_o-\Delta t_j>t_o$. Hence, from the statement 1 and 2 of lemma 3.3, $|G_i(t_o+\Delta t_j, c_o, z_o)-G_i(t_o, c_o, z_o)| \le |G_i(t_o+\Delta t_j, c_o, z_o)-G_i(t_o, c_o, z_o-\Delta t_j)|+|G_i(t_o, c_o, z_o-\Delta t_j)-G_i(t_o, c_o, z_o)|<\epsilon_2$. This together with (5.6) imply $G_i(t_o+\Delta t_j, c_o, z_o)<G_i(t_o, c_o, z_o)+\epsilon_1=g(t_o, i+1, c_o)+\epsilon_1<g(t_o+\Delta t_j, i+1, c_o)$. This, however, contradicts that $g(t_o+\Delta t_j, i+1, c_o)=\underset{t_o+\Delta t_j\le z\le T^*}{\text{Inf}} G_i(t_o+\Delta t_j, c_o, z)$. Hence (1.B), cannot be true.

Combining (1.A) and (1.B), it is obvious that no such subsequence, $\{\Delta t_{i_k}\}$ with all $\Delta t_{i_k}>0$ can exist.

CASE 2: all $\Delta t_{i_k}<0$:

The proof is omitted. However, following the same argument as in case 1, it can be shown that no such subsequence, $\{\Delta t_{i_k}\}$, with all $\Delta t_{i_k}<0$ can exist.

From case 1 and 2 it is obvious that no sequence, $\{\Delta t_i\}$, satisfying (5.2) can exist. Hence, the theorem must be true for $n=i+1$. From mathematical induction, this implies that the theorem is true.

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A MATHEMATICAL PROGRAMMING APPROACH TO IDENTIFICATION AND OPTIMIZATION OF A CLASS OF UNKNOWN SYSTEMS*

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ABSTRACT

There exists a class of decision problems for which: (1) models of input-output response functions are not available in a closed-form, functional representation; (2) informational costs associated with learning about the response function are significant. For these problems, combining identification with optimization using mathematical programming is potentially attractive. Three approaches to the identification-optimization problem are proposed: an outer-linearized approximation using relaxation (OLR); an inner-linearized approximation using restriction (ILR); and a sequential combination of inner- and outer-linearized subproblems (SIO). Algorithms based on each approach are developed and computational experience reported.

1. INTRODUCTION

There exists a class of decision problems which can be represented by the following optimization problem:

(1.1) $\underset{\mathbf{x}}{\text{Maximize}}\ f(\mathbf{x}, \mathbf{y})$,

subject to:

1. $\mathbf{x} \epsilon X$
2. $\mathbf{y} = \mathbf{s}(\mathbf{x})$

where

f is a real, single-valued function,

$\mathbf{s}$ is a real, vector-valued function, unknown to the decision maker,

$\mathbf{x}$ is an n dimensional vector,

$\mathbf{y}$ is an m dimensional vector, and

X is a convex set.

Problem (1.1) is typical of those faced by decision makers who deal with complex systems in which a well defined objective is a function of the output from a response (or production) function which is not available in a closed-form, functional representation. The unknown system response function may be a chemical reactor, an industrial manufacturing plant or a complex computer simulation model of a process. The salient characteristic of the decision problem is that $\mathbf{s}$ is unknown and informational costs associated with learning about $\mathbf{s}$ significant.

We assume that $\mathbf{y}$ is observable and unique for a given $\mathbf{x}$. Ideally the cost of experimentation should

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be included directly in the problem formulation. Unfortunately, experimental cost is virtually always a function of the number of experiments and therefore cost of experimentation cannot be expressed as a function of $\mathbf{x}$. Even so, it is clear that the algorithmic procedure used to solve (1.1) can affect the "overall" value attained in the decision problem because it will affect the number of experiments on $\mathbf{s}$ required to achieve a given value for f. Therefore, it is important to develop an algorithmic procedure for (1.1) which minimizes experimental costs by converging rapidly in terms of the number of experiments on $\mathbf{s}$.

Existing optimization techniques can be used to solve (1.1) by interrogating the system each time information is required. The fact that $\mathbf{s}$ is unknown coupled with a significant cost associated with obtaining information suggests that the question of system identification be addressed explicitly. Instead of discarding information once it is used to make a local improvement, experiments can be used to build up a model for $\mathbf{s}$ which can be used directly in the optimization process. Identification of the system in this manner may also contribute toward an improved understanding of relationships among system variables—an important goal in the study of complex systems. Further, since identification arises in conjunction with optimization, the possibility of proceeding with these two processes simultaneously arises. That is, instead of the traditional sequential approach of system description (identification) followed by optimization, the two processes can be undertaken together. This is appealing because the optimization process can interact with the identification process so that efforts are concentrated in regions relevant to good performance. Therefore, one promising approach to (1.1) involves combining identification with optimization.

To utilize the theory of concave programing, (1.1) must satisfy appropriate conditions of concavity. Sufficient conditions for (1.1) to be a concave programing problem are given below:

PROPOSITION 1.1: If $\mathbf{s}$ is partitioned into $\mathbf{s}_1$ and $\mathbf{s}_2$ which are concave and convex, respectively, on the convex set X and if f is a concave function on R^{n+m} increasing in $\mathbf{y}_1$ and decreasing in $\mathbf{y}_2$, then (1.1) is equivalent to a concave programing problem.

PROOF: Since f is increasing in $\mathbf{y}_1$ and decreasing in $\mathbf{y}_2$, (1.1) can be reformulated as:

$$\text{(1.2)} \qquad \max_{\mathbf{x},\mathbf{y}} f(\mathbf{x},\mathbf{y}),$$

subject to:

1. $\mathbf{x} \in X$
2. $\mathbf{y}_1 \leqslant \mathbf{s}_1(\mathbf{x})$
3. $\mathbf{y}_2 \geqslant \mathbf{s}_2(\mathbf{x})$.

Problem (1.2) is a concave programing problem since f is concave on R^{n+m} and the constraints form a convex set. Q.E.D.

The algorithms described below are based on the conditions specified in Proposition (1.1) and the concept of combining identification with optimization in an iterative procedure. Two alternative representations for concave (convex) functions, outer linearization and inner linearization, are considered in this paper. When these representations are used as a basis for identification they lead naturally to solution strategies of relaxation and restriction respectively (although not yet in common use this terminology is introduced in [5] and used here for its descriptive convenience). Section 2 presents an algorithm based on an outer-linearized representation of $\mathbf{s}$, section 3 considers an inner-linearized representation, and section 4 a direct combination of inner and outer linearization. Computational experience with these three approaches is reported in section 5.

2. THE APPROACH BASED ON AN OUTER-LINEARIZED REPRESENTATION FOR s (OLR)

A concave (convex) function can be approximated as closely as desired by outer linearization over an arbitrarily dense grid of points. Such an approximation never under (over) estimates the function and requires the identification of appropriate linear supporting functions. We assume that a subgradient, γ_i, and hence a linear support, $s_i(\bar{\mathbf{x}}) + \gamma_i(\mathbf{x} - \bar{\mathbf{x}})$, exists for each component of $\mathbf{s}$ for all $\bar{\mathbf{x}} \epsilon X$. The strategy of relaxation (see [5] for a more detailed discussion of both outer linearization and relaxation) provides a means for selectively identifying those subgradients of $\mathbf{s}$ which are important to the optimization process.* It consists of solving a sequence of "relaxed" problems in which some set of constraints is removed. Those constraints which are removed are evaluated at the solution to each relaxed problem. If they are satisfied, an optimum solution has been found. If not, one or more of the violated constraints are added to the set which is included in the relaxed problem and the problem resolved. The procedure is repeated until all constraints are satisfied.

The idea behind this approach has been used often in mathematical programing and stems from the cutting plane method of Kelly [9] and Cheney and Goldstein [1]. Although the resulting algorithm is basically a cutting plane algorithm, the motivation and subproblems (f is not required to be outer-linearized) are sufficiently different to justify the development presented below.

If (1.1) is reformulated as described in Proposition (1.1) it can be written:†

(2.1) $$\underset{\mathbf{x},\mathbf{y}}{\text{Maximize}}\ f(\mathbf{x}, \mathbf{y}),$$

subject to:

1. $\mathbf{x} \epsilon X$
2. $\mathbf{y} \leqq \mathbf{s}(\mathbf{x})$

Using an outer-linearized approximation of $\mathbf{s}$ over an arbitrarily fine grid of points, in principle (2.1) can be reformulated as:

(2.2) $$\underset{\mathbf{x},\mathbf{y}}{\text{Maximize}}\ f(\mathbf{x}, \mathbf{y}),$$

subject to:

1. $\mathbf{x} \epsilon X$
2. $\mathbf{y} \leqq \mathbf{s}(\bar{\mathbf{x}}) + [\bar{\gamma}](\mathbf{x} - \bar{\mathbf{x}})$ all $\bar{\mathbf{x}} \epsilon X$

where $[\bar{\gamma}] = [\bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_m]^t$ is a matrix of subgradients of $\mathbf{s}$ at $\bar{\mathbf{x}}$.

We say "in principle" because by using the strategy of relaxation we are never required to know a subgradient at every $\bar{\mathbf{x}} \epsilon X$. Rather, at each iteration only a subset of constraints are included. If we denote the set of constraints which are included as those corresponding to supports at $\mathbf{x}^j$, $j \epsilon P$ where $\mathbf{x}^j \epsilon X$, the relaxed version of (2.2) is:

(OR) $$\underset{\mathbf{x},\mathbf{y}}{\text{Maximize}}\ f(\mathbf{x}, \mathbf{y}),$$

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*For a discussion of the additional mild conditions on $\mathbf{s}$ which guarantee the existence of a subgradient, see, for example, [6].

†As shown in Proposition 1.1, (1.1) is a concave programing problem if f is decreasing in y_i and s_i is convex. For notational ease, we adopt the convention that, if s_i is convex, a change of variable $y_i' = -y_i$ and $s_i' = -s_i$ is introduced so that $\mathbf{s}$ remains concave and f increasing in $\mathbf{y}$.

subject to:

1. $\mathbf{x} \epsilon X$
2. $\mathbf{y} \leq \mathbf{s}(\mathbf{x}^j) + [\gamma^j](\mathbf{x} - \mathbf{x}^j), j \epsilon P.$

If the solution to (OR), $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, is feasible in (2.2), it is optimal. If it is not feasible, a violated constraint is added to set P and (OR) resolved. A natural criterion is to add the most violated constraint. It is easy to demonstrate that if $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is not feasible in (2.2), then the set of constraints $\mathbf{y} \leq \mathbf{s}(\bar{\mathbf{x}}) + [\bar{\gamma}](\mathbf{x} - \bar{\mathbf{x}})$ is the most violated set of constraints in the sense that for each component of $\mathbf{y}$

$$[\bar{y}_i - s_i(\bar{\mathbf{x}})] = \underset{x \epsilon X}{\text{Maximum}} \{\bar{y}_i - [s_i(\mathbf{x}) + \gamma_i(\bar{\mathbf{x}} - \mathbf{x})]\}. \tag{2.3}$$

From (2.3) we also have an optimality check since if $\bar{\mathbf{y}} \leq \mathbf{s}(\bar{\mathbf{x}})$ it is feasible in (2.2).

Therefore, the solution to relaxed problem (OR) identifies the set of most violated constraints. Experimentation on the system to determine a linear support for $\mathbf{s}$ at $\bar{\mathbf{x}}$ will produce the entire matrix $[\bar{\gamma}]$ and therefore, the most violated constraint for each component of $\mathbf{s}$ can be added to set P at each iteration. The algorithm described below specifies the iterative procedure based on using outer linearization in combination with relaxation for this problem.

Algorithm Based on Outer Linearization and Relaxation (OLR)

STEP 0: Find a feasible initial solution, $\mathbf{x}^0 \epsilon X$, $\mathbf{y}^0 = \mathbf{s}(\mathbf{x}^0)$; initialize set $\mathbf{x}^j$, $j \epsilon P$ with the null set. Substitute $(\mathbf{x}^0, \mathbf{y}^0)$ into f to obtain lower bound $f(\mathbf{x}^0, \mathbf{y}^0)$.

STEP 1: Find a linear support for $\mathbf{s}(\mathbf{x})$ at $\mathbf{x}^0$ and add $\mathbf{x}^0$ to set $\mathbf{x}^j$, $j \epsilon P$. Find an optimal solution, $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ to (OR); the optimal value is an upper bound. Check the difference between upper and lower bounds; if less than a predetermined tolerance, terminate. Find $\bar{\bar{\mathbf{y}}} = \mathbf{s}(\bar{\mathbf{x}})$; determine the lower bound $f(\bar{\mathbf{x}}, \bar{\bar{\mathbf{y}}})$; and check for termination as above. Go to Step 2.

STEP 2: Set $\mathbf{x}^0 = \bar{\mathbf{x}}$, $\mathbf{y}^0 = \bar{\bar{\mathbf{y}}}$ and return to Step 1.

As indicated above (OLR) is a cutting plane algorithm and differs from the method suggested by Kelly and by Cheney and Goldstein only in that the relaxed problems are not in general linear. The convergence of this method to the optimal value of equivalent problems (2.2) and (2.1) is well known (see, for example [9] or [11]).* Clearly the procedure results in identifying $\mathbf{s}$ in terms of an outer-linearized approximation for $\mathbf{s}$. At each iteration this approximation is used in conjunction with the optimization process to determine the location of the next set of linear supports to be added to the approximation. Finally, when the approximation is good enough to satisfy a termination criterion the algorithm stops.

3. THE APPROACH BASED ON AN INNER-LINEARIZED REPRESENTATION FOR s (ILR)

A convex set or concave (convex) function can be approximated as closely as desired by inner linearization over an arbitrarily dense grid of points. Such an approximation is always conservative in that no points outside the convex set are included and the concave function is never overestimated. The strategy of restriction (see [5] for a more detailed discussion of inner linearization and restriction) provides a means for determining those values of $\mathbf{s}$ which are important to the optimization process. It

*If we add the restriction that X is compact, algorithm (OLR) can be shown to converge using Zangwill's Cutting Plane Convergence Theorem following the proof given for the Concave-Cutting-Plane Algorithm ([11], chap. 14).

consists of solving a sequence of problems with a subset of the variables restricted to zero, hence temporarily removing them from the problem. After each restricted problem is solved, the solution is tested for optimality in the original problem. If it is not optimal, some subset of restricted variables is "released" and the procedure repeated. This approach has also been used previously in mathematical programing and underlies the classic Dantzig-Wolfe decomposition procedure ([2] and [3]).

If the convex set X and the concave function $\mathbf{s}$ are inner-linearized over an arbitrarily fine base of grid points $\langle \mathbf{x}^1, \mathbf{x}^2, \ldots, \rangle$ in X, (1.1) is (in principle at least) equivalent to:

$$\text{(3.1)} \qquad \underset{\alpha}{\text{Maximize}}\, f\left(\sum_j \alpha^j \mathbf{x}^j, \sum_j \alpha^j \mathbf{s}(\mathbf{x}^j)\right),$$

subject to:

		Multipliers
1.	$\sum_j \alpha^j = 1$	π
2.	$\alpha^j \geq 0$	λ.

Formulation (3.1) is conceptual in nature since we do not propose to evaluate $\mathbf{s}$ at every grid point and in fact will not even specify the base explicitly. Therefore we can stipulate that the solution to (1.1) as well as any other points we might consider are included and hence (3.1) and (1.1) are equivalent.

The strategy of restriction specifies that (3.1) be solved with a subset of the constraints $\alpha^j \geq 0$ forced to hold with equality. If we force a subset of constraints α^j, $j \notin P$ to hold with equality, where set $\mathbf{x}^j$, $j \in P$ is a nonempty set consisting of grid points at which $\mathbf{s}(\mathbf{x}^j)$ is known, we have a restricted version of (3.1):

$$\text{(IR)} \qquad \underset{\alpha}{\text{Maximize}}\, f\left(\sum_j \alpha^j \mathbf{x}^j, \sum_j \alpha^j \mathbf{s}(\mathbf{x}^j)\right)$$

subject to:

		Multipliers
1.	$\sum_j \alpha^j = 1$	π
2.	$\alpha^j \geq 0 \quad j \in P$	λ.
3.	$\alpha^j = 0 \quad j \notin P$	

Now, since $\alpha^j = 0$ for $j \notin P$, $\mathbf{s}(\mathbf{x}^j)$ is not required for $j \notin P$ and the base only needs to be specified explicitly for a subset of points.

PROPOSITION 3.1: If f is differentiable and $\hat{\alpha}$ the solution to (IR), optimal multipliers $\hat{\lambda}^j$ exist and are given by:

$$\text{(3.2)} \qquad \hat{\lambda}^j = -\hat{\pi} - [\nabla_x f(\mathbf{x}', \mathbf{y}')][\mathbf{x}^j] - [\nabla_y f(\mathbf{x}', \mathbf{y}')][\mathbf{s}(\mathbf{x}^j)]$$

where $\mathbf{x}' = \sum_j \hat{\alpha}^j \mathbf{x}^j$, $\mathbf{y}' = \sum_j \hat{\alpha}^j \mathbf{s}(\mathbf{x}^j)$ and $\hat{\pi}$ is an optimal multiplier.

PROOF: Expression (3.2) follows directly from the Kuhn-Tucker conditions. Q.E.D.

PROPOSITION 3.2: If f is differentiable, $\hat{\alpha}$, the solution to (IR), is optimal in (3.1) (and hence $\mathbf{x}' = \sum_j \hat{\alpha}^j \mathbf{x}^j$ in (1.1)) if

$$\bar{\lambda} = -\underset{x \in X}{\text{Maximum}} \{\hat{\pi} + [\nabla_x f(\mathbf{x}', \mathbf{y}')]\ [\mathbf{x}] + [\nabla_y f(\mathbf{x}', \mathbf{y}')]\ [\mathbf{s}(\mathbf{x})]\}. \tag{3.3}$$

is nonnegative.

PROOF: The Kuhn-Tucker conditions for (3.1) and (IR) are identical except that $\hat{\lambda}^j \geq 0$ all j in (3.1). If $\bar{\lambda} \geq 0$, then (3.2) must be nonnegative for all j and therefore $(\hat{\alpha}, \hat{\lambda}, \hat{\pi})$ satisfy the Kuhn-Tucker conditions for (3.1). Q.E.D.

Problem (3.3) is a well defined concave programing problem since f is increasing in $\mathbf{y}$ and $\mathbf{s}$ is concave. If the value of (3.3) is negative, then there exists a negative multiplier for at least one constraint in (IR). Since any multiplier λ^k is the negative of a subderivative of the optimal value of (IR) with respect to the right hand side of the corresponding constraint (see, for example, [6] or [10]), a negative multiplier indicates that the optimal value may be increased if α^k is increased from the zero level. The strategy of restriction specifies that a constraint corresponding to some $\lambda^k < 0$ be removed from the equality set and added to set P, which allows the associated α^k to achieve a positive level. This is accomplished by adding $\mathbf{x}^k$ to the grid of points $\mathbf{x}^j$, $j \in P$.

A natural criterion for choosing constraints to be removed from the equality set is the most negative multiplier criterion.

PROPOSITION 3.3: If f is differentiable and $\mathbf{x}' = \sum_j \hat{\alpha}^j \mathbf{x}^j$, $\mathbf{y}' = \sum_j \hat{\alpha}^j \mathbf{s}(\mathbf{x}^j)$ is a solution to (IR), then the multiplier associated with any $\mathbf{x} \in X$ is unique and the solution to (3.3) yields the most negative multiplier for $\mathbf{x} \in X$.

PROOF: Clearly the solution to (3.3), $\bar{\mathbf{x}}$, is independent of $\hat{\pi}$ and $\bar{\lambda}$ is the most negative multiplier associated with any constraint in (IR) consistent with $\hat{\pi}$, $\mathbf{x}'$, and $\mathbf{y}'$. The optimal multiplier $\hat{\pi}$ satisfies the Kuhn-Tucker conditions for (IR):

1. $\hat{\alpha}$ feasible in (IR).
2. $\hat{\lambda}_j \geq 0\ j \in P$.
3. $\hat{\pi} + [\nabla_x f(\mathbf{x}', \mathbf{y}')] \cdot [\mathbf{x}^j] + [\nabla_y f(\mathbf{x}', \mathbf{y}')] \cdot [\mathbf{s}(\mathbf{x}^j)] + \hat{\lambda}^j = 0$, all j.
4. $\sum_{j \in P} \hat{\lambda}^j \hat{\alpha}^j = 0$.

By multiplying 3 above by $\hat{\alpha}^j$, summing over j, and using conditions 1 and 4, we have

$$\hat{\pi} = -[\nabla_x f(\mathbf{x}'\mathbf{y}')] \cdot [\mathbf{x}'] - [\nabla_y f(\mathbf{x}', \mathbf{y}')] \cdot [\mathbf{y}'].$$

Hence $\hat{\pi}$ and consequently all multipliers are unique given $(\mathbf{x}', \mathbf{y}')$. Q.E.D.

Problem (3.3) is then the "pricing" problem for (IR) using the most negative multiplier criterion and provides a means for selecting points to be explicitly evaluated in our grid (i.e., added to $\mathbf{x}^j$, $j \in P$) which of course adds to the inner-linearized approximation for $\mathbf{s}$. However, since $\mathbf{s}$ is unknown, (3.3) cannot be solved directly. We require an alternate representation for $\mathbf{s}$ which is useful in the pricing problem. It is easy to show that an inner-linearized approximation for $\mathbf{s}$ in (3.3) over the convex hull of known points cannot identify a point corresponding to a negative multiplier. If an outer-linearized approximation is used for $\mathbf{s}$ at some set of points $\mathbf{x}^j$, $j \in \bar{P}$, where subgradients have been previously found, (3.3) becomes:

$$\underset{\mathbf{x},\mathbf{y}}{\text{Maximize}} \{\hat{\pi} + [\nabla_x f(\mathbf{x}', \mathbf{y}')] \cdot [\mathbf{x}] + [\nabla_y f(\mathbf{x}', \mathbf{y}')] \cdot [\mathbf{y}]\}, \tag{3.4}$$

subject to:

1. $\mathbf{x} \epsilon X$
2. $\mathbf{y} \leqq \mathbf{s}(\mathbf{x}^j) + [\gamma^j](\mathbf{x} - \mathbf{x}^j), j \epsilon \bar{P}$.

Problem (3.4) can identify a point which is associated with a negative multiplier, and we propose using it as the pricing problem for (IR).* It would be nice if it could be shown that using (3.4) in this manner identifies a negative multiplier in (IR) if one exists. Unfortunately, the outer-linearized approximation for $\mathbf{s}$ may result in an optimal value for (3.4) which is positive when the multiplier associated with the solution (3.4) is not negative. (In computational experience to date, except for the choice of the first point to add to grid $\mathbf{x}^j, j \epsilon P$ – initialized with only a single point – (3.4) has never failed to identify a negative multiplier when one exists.) However, since the value of (3.4) is an upper bound on (3.3), from Proposition 3.2 it is seen that, if (3.4) is nonpositive, the solution to (IR) is optimal in (3.1) (and (1.1)). Furthermore, as shown following specification of algorithm (ILR), using (3.4) as the pricing problem in conjuction with (ILR) results in identifying a negative multiplier if one exists.

For notational convenience we difine an $n+m$ dimensional vector $\mathbf{z} \underline{\underline{\Delta}} (\mathbf{x}, \mathbf{y})$. The solution to (IR) at the kth iteration can then be expressed as

$$\mathbf{z}^k = \left\{ \sum_{j \epsilon P^k} (\hat{\alpha}^j)^k \mathbf{x}^j, \sum_{j \epsilon P^k} (\hat{\alpha}^j)^k \mathbf{s}(\mathbf{x}^j) \right\}.$$

PROPOSITION 3.4: If the solution to $(\mathrm{IR})^k$, $\mathbf{z}^k$, is not an optimal solution to (3.1), then $f(\mathbf{z}^{k+1}) > f(\mathbf{z}^k)$ if, and only if, $\bar{\mathbf{x}}^k$ associated with a negative multiplier is added to set $\mathbf{x}^j, j \epsilon P^k$ creating set $\mathbf{x}^j, j \epsilon P^{k+1}$ for $(\mathrm{IR})^{k+1}$.

PROOF: If $\bar{\mathbf{x}}^k$ is added to set $\mathbf{x}^j, j \epsilon P^k$ creating P^{k+1} for $(\mathrm{IR})^{k+1}$ then since $P^k \subset P^{k+1}$, $\mathbf{z}^k$ is feasible in $(\mathrm{IR})^{k+1}$ and $f(\mathbf{z}^{k+1}) \geqq f(\mathbf{z}^k)$. Now assume $f(\mathbf{z}^{k+1}) = f(\mathbf{z}^k)$. In this case $\mathbf{z}^k$ is not only feasible in $(\mathrm{IR})^{k+1}$, but an optimal solution. From the Kuhn-Tucker conditions, for $(\mathrm{IR})^{k+1}$, $\lambda^j \geqq 0, j \epsilon P^{k+1}$ and since $\bar{\mathbf{x}}^k = \mathbf{x}^j$ for some $j \epsilon P^{k+1}$, $\bar{\lambda}^k \geqq 0$. But by hypothesis $\bar{\lambda}^k < 0$ for solution $\mathbf{z}^k$ and from Proposition 3.3 we know that $\bar{\lambda}^k$ is unique for $\mathbf{z}^k$. Therefore, we have a contradiction and $\mathbf{z}^k$ cannot be a solution to $(\mathrm{IR})^{k+1}$. Therefore, $f(\mathbf{z}^{k+1}) > f(\mathbf{z}^k)$. The converse follows trivially from the definition of a negative multiplier. Q.E.D.

Algorithm Based on Inner Linearization and Restriction (ILR)

STEP 0: Find a feasible initial solution, $\mathbf{x}^0 \epsilon X$, $\mathbf{y}^0 = \mathbf{s}(\mathbf{x}^0)$; initialize $\mathbf{x}^j, j \epsilon P$ with the point $\mathbf{x}^0$ and $\mathbf{x}^j, j \epsilon \bar{P}$ with the null set. Set $\mathbf{x}' = \mathbf{x}^0$ and $\mathbf{y}' = \mathbf{y}^0$ and substitute $(\mathbf{x}^0, \mathbf{y}^0)$ into f to obtain lower bound $f(\mathbf{x}^0, \mathbf{y}^0)$.

STEP 1: Find a linear support for $\mathbf{s}(\mathbf{x})$ at $\mathbf{x}^0$ and add $\mathbf{x}^0$ to set $\mathbf{x}^j, j \epsilon \bar{P}$. Find an optimal solution, $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, to the outer-linearized, most negative multiplier problem (3.4). If the optimal value of (3.4) is nonpositive, stop. Otherwise, since the optimal value of (3.4) provides an upper bound to (1.1) when the constant term $f(\mathbf{x}', \mathbf{y}')$ is added, check the difference between upper and lower bounds; if less than a

*Other approaches to the negative multiplier problem are also possible. For instance, a one-dimensional optimization could be performed between $\mathbf{x}'$ and $\bar{\mathbf{x}}$; however, because an objective of the algorithm is rapid convergence of the overall problem in terms of the number of evaluations of $\mathbf{s}$, a question arises concerning the accuracy with which such one-dimensional subproblems should be solved. Alternative approaches to (3.3) are not pursued here.

predetermined tolerance, terminate. Find $\bar{y}=s(\bar{x})$; add $\bar{x}$ to set x^j, $j \in P$; determine lower bound, $f(\bar{x}, \bar{y})$; and check for termination as above. Go to Step 2.

STEP 2: Find an optimal solution, $\left(x' = \sum_{j \in P} \hat{\alpha}^j x^j, y' = \sum_{j \in P} \hat{\alpha}^j s(x^j)\right)$, to problem (IR). (If the optimal value of (IR) does not increase for this iteration set (x', y') equal to the optimal solution for the previous iteration.) The optimal value of (IR) is a lower bound; check for termination as in Step 1. Set $x^0 = \bar{x}$ and $y^0 = \bar{y}$ and return to Step 1.

PROPOSITION 3.5: Algorithm (ILR) will identify a negative multiplier if one exists.

PROOF: If $f(z^{k+1})$ is strictly increasing in k, $\bar{x}^k$ is a negative multiplier from Proposition 3.4. If $f(z^{k+1}) = f(z^k)$ in Step 2, the algorithm resolves (3.4) after adding a new set of constraints. But (3.4) is just a relaxed, outer-linearized approximation to (3.3) and under these conditions the algorithm is seen to be an application of (OLR) to (3.3). Therefore, at least in the limit, a negative multiplier will be identified. Q.E.D.

As shown below, theoretical convergence of this procedure can be established if the most negative multiplier problem, (3.3) is used as the pricing problem instead of (3.4). Therefore, by changing to a mixed algorithm which utilizes (3.4) for a large, but finite number of iterations and then reverts to (3.3), theoretical convergence can be guaranteed. (Note that (3.3) can be solved even though s is unknown. It is, however, likely to be expensive in terms of experimental cost.) As a practical matter, the motivation for combining identification with optimization is to increase the rate of convergence in terms of experimental observations and termination criteria will rely on a comparison between experimental cost and the difference between upper and lower bounds. Therefore, primary interest involves performance during the initial stages of the iterative procedure. Section 5 summarizes computational investigations of this aspect of the algorithms.

PROPOSITION 3.6: If f is continuously differentiable on R^{n+m}, s is concave and subgradients exist for all $x \in X$, X is convex and a sequence $\{z^k\}^\infty$ is generated by (ILR), there cannot be a subsequence $\{z^k\}k \in K$ with the following three properties:

1. $z^k \to z^\infty \quad k \in K$;
2. $\bar{x}^k \to \bar{x}^\infty \quad k \in K$;
3. $[\nabla_x f(z^\infty)][\bar{x}^\infty - x^\infty] + [\nabla_y f(z^\infty)][s(\bar{x}^\infty) - y^\infty] > 0$.

PROOF: From 3 there is a $\beta > 0$ such that

$$[\nabla_x f(z^\infty)][\bar{x}^\infty - x^\infty] + [\nabla_y f(z^\infty)][s(\bar{x}^\infty) - y^\infty] = \beta > 0.$$

Since f is continuously differentiable and s is continuous, there must be a $k' \in K$ sufficiently large so that for any $k \geq k'$.

$$[\nabla_x f(z^k)][\bar{x}^k - x^k] + [\nabla_y f(z^k)][s(\bar{x}^k) - y^k] > \beta/2 \tag{3.5}$$

and a $\delta > 0$ exists such that for $0 \leq \tau \leq \delta$.

$$[\nabla_x f(z^k + \tau(\bar{x}^k - x^k, s(\bar{x}^k) - y^k))][\bar{x}^k - x^k] + [\nabla_y f(z^k + \tau(\bar{x}^k - x^k, s(\bar{x}^k) - y^k))][s(\bar{x}^k) - y^k] > \beta/4 \tag{3.6}$$

Since both $\mathbf{z}^k = (\mathbf{x}^k, \mathbf{y}^k)$ and $(\bar{\mathbf{x}}^k, \mathbf{s}(\bar{\mathbf{x}}^k))$ are feasible in $(\mathrm{IR})^{k+1}$ and since the feasible region is convex, there must exist some $0 < \tau^* \leq \delta$ such that $\mathbf{z}^k + \tau^*(\bar{\mathbf{x}}^k - \mathbf{x}^k, \mathbf{s}(\bar{\mathbf{x}}^k) - \mathbf{y}^k)$ is feasible in $(\mathrm{IR})^{k+1}$. Consequently,

$$f(\mathbf{z}^{k+1}) \geq f(\mathbf{z}^k + \tau^*(\bar{\mathbf{x}}^k - \mathbf{x}^k, \mathbf{s}(\bar{\mathbf{x}}^k) - \mathbf{y}^k), \tag{3.7}$$

and by use of Taylor's expansion

$$f(\mathbf{z}^{k+1}) \geq f(\mathbf{z}^k) + [\nabla_x f(\mathbf{z}^0)][\tau^*(\bar{\mathbf{x}}^k - \mathbf{x}^k)] + [\nabla_y f(\mathbf{z}^0)][\tau^*(\mathbf{s}(\bar{\mathbf{x}}^k) - \mathbf{y}^k)], \tag{3.8}$$

where

$$\mathbf{z}^0 = \mathbf{z}^k + \theta\tau^*(\bar{\mathbf{x}}^k - \mathbf{x}^k, \mathbf{s}(\bar{\mathbf{x}}^k) - \mathbf{y}^k), 0 \leq \theta \leq 1.$$

Now, since $0 \leq \theta\tau^* \leq \delta$.

$$f(\mathbf{z}^{k+1}) \geq f(\mathbf{z}^k) + \tau^*\beta/4 \tag{3.9}$$

Because $f(\mathbf{z}^k)$ is continuous and monotonic we have

$$\lim_{k \to \infty} f(\mathbf{z}^k) = f(\mathbf{z}^\infty).$$

Therefore, for k sufficiently large we have

$$f(\mathbf{z}^\infty) \leq f(\mathbf{z}^k) + \tau^*\beta/8. \tag{3.10}$$

For $k \geq k'$ and sufficiently large so that (3.10) holds we have from (3.9) and (3.10)

$$f(\mathbf{z}^{k+1}) > f(\mathbf{z}^\infty).$$

Since $f(\mathbf{z}^k)$ is monotonic we have a contradiction and hence all three properties cannot hold simultaneously. Q.E.D.

PROPOSITION 3.7: If f is concave, increasing in $\mathbf{y}$ and continuously differentiable on R^{n+m}; $\mathbf{s}$ is concave and subgradients exists for all $\mathbf{x} \in X$; and X is compact, then if (3.3) is substituted for (3.4) and the termination tolerance is set to zero (ILR) converges to the optimal solution to (1.1).

PROOF: The equivalence of (1.1) and (3.1) was previously argued, therefore if the algorithm converges to a solution for (3.1) it converges to a solution for (1.1).

1. If the algorithm terminates, it terminates at a solution since if the solution to (3.3) is zero the Kuhn-Tucker conditions for (3.1) are satisfied.

2. Assume the algorithm generates an infinite sequence of points none of which is a solution. Since it is easy to construct a compact set $Z \underline{\Delta} \{\mathbf{x}, \mathbf{y} | \mathbf{x} \in X, M \leq \mathbf{y} \leq \mathbf{s}(\mathbf{x})\}$, such that $\mathbf{z}^k \in Z$ all k, there

must exist a convergent subsequence:

$$\mathbf{z}^k \to \mathbf{z}^\infty \qquad k \epsilon K.$$

Assume that $\mathbf{z}^\infty$ is not a solution. Since X is compact there exists a $K' \subset K$ such that

$$\bar{\mathbf{x}}^k \to \bar{\mathbf{x}}^\infty \qquad k \epsilon K' \tag{3.11}$$

and

$$\mathbf{z}^k \to \mathbf{z}^\infty \qquad k \epsilon K'. \tag{3.12}$$

Consider now the most negative multiplier problem $(3.3)^\infty$:

$$\sigma^* = \underset{\mathbf{x} \epsilon X}{\text{Maximum}} \; [\nabla_x f(\mathbf{z}^\infty)][\mathbf{x} - \mathbf{x}^\infty] + [\nabla_y f(\mathbf{z}^\infty)][\mathbf{s}(\mathbf{x}) - \mathbf{y}^\infty] \tag{3.13}$$

where $\hat{\pi}$ has been replaced by the expression developed in Proposition 3.3. If $\mathbf{z}^\infty$ is not a solution, then a negative multiplier exists and Proposition 3.3 guarantees that the solution to (3.13), $\bar{\mathbf{x}}^*$, is such that $\sigma^* > 0$. Since f is continuously differentiable there must exist a $k' \epsilon K'$ such that for $k \geq k'$

$$[\nabla_x f(\mathbf{z}^k)][\bar{\mathbf{x}}^* - \mathbf{x}^k] + [\nabla_y f(\mathbf{z}^k)][\mathbf{s}(\bar{\mathbf{x}}^*) - \mathbf{y}^k] > \sigma^*/2.$$

Therefore, since $\bar{\mathbf{x}}^* \epsilon X$ is feasible in $(3.3)^k$, the

$$\underset{\mathbf{x} \epsilon X}{\text{Maximum}} \; [\nabla_x f(\mathbf{z}^k)][\mathbf{x} - \mathbf{x}^k] + [\nabla_y f(\mathbf{z}^k)][\mathbf{s}(\mathbf{x}) - \mathbf{y}^k]$$

yields a solution $\bar{\mathbf{x}}^k$ such that

$$[\nabla_x f(\mathbf{z}^k)][\mathbf{x}^k - \mathbf{x}^k] + [\nabla_y f(\mathbf{z}^k)][\mathbf{s}(\bar{\mathbf{x}}^k) - \mathbf{y}^k] > \sigma^*/2 \tag{3.14}$$

Since f is continuously differentiable and s continuous, taking the limit

$$\lim_{k \epsilon K'} \{[\nabla_x f(\mathbf{z}^k)][\bar{\mathbf{x}}^k - \mathbf{x}^k] + [\nabla_y f(\mathbf{z}^k)][\mathbf{s}(\bar{\mathbf{x}}^k) - \mathbf{y}^k]\} \geq \sigma^*/2$$

yields

$$[\nabla_x f(\mathbf{z}^\infty)][\bar{\mathbf{x}}^\infty - \mathbf{x}^\infty] + [\nabla_y f(\mathbf{z}^\infty)][\mathbf{s}(\bar{\mathbf{x}}^\infty) - \mathbf{y}^\infty] \geq \sigma^*/2 > 0. \tag{3.15}$$

But Proposition 3.6 established that conditions (3.15), (3.11), and (3.12) cannot all exist. Therefore we have a contradiction and the limit of any convergent subsequence is a solution. Q.E.D.

The procedure is based on identifying s in terms of an inner-linearized approximation. However, the pricing problem requires an outer-linearized approximation, and so both approximations are obtained and utilized in the algorithm. When these approximations are good enough to satisfy a termination criterion the algorithm stops.

4. THE APPROACH BASED ON A SEQUENTIAL COMBINATION OF INNER AND OUTER LINEARIZATION (SIO)

It is natural to consider combining the two approaches discussed above so that both inner- and outer-linearized approximations are built up as efficiently as possible. One appealing approach is to alternate between the relaxed, outer-linearized problem (OR), and the restricted, inner-linearized problem (IR). The solution to (IR) can be used to refine the supporting plane approximation to **s** and (OR) to identify grid refinement points. This combination is motivated by noting that replacing (3.4) by (OR) in (ILR) offers potential for a significantly improved upper bound as well as more rapid convergence by adding grid points based on the best current outer-linearized approximation to (1.1). Similarly from the point of view of (OLR) using (IR) for the support refinement problem can identify two increasing lower bounds.

Algorithm Based on Sequential Combination of Inner- and Outer-Linearized Problems (SIO)

STEP 0: Find a feasible initial solution, $\mathbf{x}^0 \epsilon X$, $\mathbf{y}^0 = \mathbf{s}(\mathbf{x}^0)$; initialize set $\mathbf{x}^j$, $j \epsilon P$ with point $\mathbf{x}^0$ and $\mathbf{x}^j$, $j \epsilon \bar{P}$ with the null set. Substitute $(\mathbf{x}^0, \mathbf{y}^0)$ into f to obtain lower bound $f(\mathbf{x}^0, \mathbf{y}^0)$.

STEP 1: Find a linear support for $\mathbf{s}(\mathbf{x})$ at $\mathbf{x}^0$ and add $\mathbf{x}^0$ to set $\mathbf{x}^j$, $j \epsilon \bar{P}$. Find an optimal solution, $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ to (OR); the optimal value is an upper bound. Check the difference between upper and lower bounds; if less than a predetermined tolerance, terminate. Find $\bar{\bar{\mathbf{y}}} = \mathbf{s}(\mathbf{x})$; add $\bar{\mathbf{x}}$ to set $\mathbf{x}^j$, $j \epsilon P$; determine lower bound, $f(\bar{\mathbf{x}}, \bar{\bar{\mathbf{y}}})$; and check for termination as above. Go to Step 2.

STEP 2: Find an optimal solution, $\left(\mathbf{x}' = \sum_{j \epsilon P} \hat{\alpha}^j \mathbf{x}^j,\ \mathbf{y}' = \sum_{j \epsilon P} \hat{\alpha}^j \mathbf{s}(\mathbf{x}^j)\right)$, to (IR). The optimal value of (IR) is a lower bound. Check to see if $\mathbf{x}' = \mathbf{x}^j$ for some $j \epsilon P$.

A. If $\mathbf{x}' \neq \mathbf{x}^j$ for $j \epsilon \bar{P}$:

Check for termination as in Step 1. Set $\mathbf{x}^0 = \mathbf{x}'$, find $\mathbf{y}^0 = \mathbf{s}(\mathbf{x}^0)$, and add $\mathbf{x}^0$ to set $\mathbf{x}^j$, $j \epsilon P$. Find lower bound $f(\mathbf{x}^0, \mathbf{y}^0)$ and check for termination as in Step 1. Return to Step 1.

B. If $\mathbf{x}' = \mathbf{x}^j$ for some $j \epsilon \bar{P}$:

Set $\mathbf{x}^0 = \bar{\mathbf{x}}$, $\mathbf{y}^0 = \bar{\bar{\mathbf{y}}}$ and return to Step 1.

PROPOSITION 4.1: Algorithm (SIO) produces three increasing lower bounds per iteration: $f(\mathbf{x}^0, \mathbf{y}^0) \geq f(\mathbf{x}', \mathbf{y}') \geq f(\bar{\mathbf{x}}, \bar{\bar{\mathbf{y}}})$

PROOF: The first inequality holds because $\mathbf{x}^0 = \mathbf{x}'$ and $\mathbf{y}^0 \geq \mathbf{y}'$ and f is increasing in $\mathbf{y}$. The second inequality holds since $f(\mathbf{x}', \mathbf{y}')$ is the solution to (IR) in which $(\bar{\mathbf{x}}, \bar{\bar{\mathbf{y}}})$ is a grid point. Q.E.D.

It can be seen that whenever (OR) does not identify a point corresponding to a negative multiplier or (IR) fails to find a violated constraint, (SIO) reverts to (OLR) until progress is made. This feature can be extended to ensure theoretical convergence by permanently reverting to (OLR) after a fixed number of iterations. Convergence could also be guaranteed by replacing (OR) by (3.3) after a fixed number of iterations as was the case with (ILR).

The structure of (SIO) provides the opportunity for modifications which are attractive when experimental costs are significant and there is a substantial cost differential between obtaining a

subgradient and evaluating s at a single point. The latter suggests that we continue to iterate (IR) at step 2 until no improvement exists in the lower bound. The former that a heuristic device be added to prevent placing a number of points or supports very close to each other, unless the point is near the optimum. These modifications were included in the computational work reported here (see [8] for details of the modification).

5. COMPUTATIONAL EXPERIENCE

The most pervasive difficulty in evaluating nonlinear programing algorithms is the infinite variety of nonlinear problems. Using the number of experiments performed on the system response function (instead of the customary criterion of time) in the measure of effectiveness eliminates consideration of the computer system and efficiency of programing in the evaluation. The variety of problems, however, prevents obtaining conclusive results in comparing computational approaches to the problem posed in this study just as it does in the ordinary nonlinear case. Nevertheless, computational experience can provide tentative conclusions and lead to valuable insights for developing more effective computational approaches.

The primary objective of the computational work was to evaluate the three computational approaches. In addition, the issue of termination criteria was tentatively explored. An optimal stopping rule determines the number of experiments which results in the value of the payoff function less the cost of experimentation achieving its maximum value for a given problem and algorithm. Such a rule would require prior knowledge of the behavior of the algorithm for the problem at hand; because such knowledge is not available, practical stopping rules must be based on estimates of this behavior. This study considered only a natural stopping rule (or extreme in the sense that the overall objective function is guaranteed to be reduced if more experiments are made): terminate when the difference between upper and lower bounds is less than the cost of obtaining a new solution.

A series of five problems were used to test the proposed algorithms. Details of all problems are given in [8]. The appendix discusses in detail Problem 1 and its solution using each of the three algorithms. Computational results for all test problems are summarized in Tables 1 through 5. They show

TABLE 1. *Difference between upper and lower bounds for problem 1*

(Optimal value = 219.2)

Number of experiments[a]	(OLR)	(ILR)	(SIO)
4	38.21	314.85	38.21
5	23.74	314.85	22.44
6	23.74	161.67	19.87
9	9.31	161.67	19.87
10	2.79	160.05	3.38
11	2.79	160.05	0.71
14	0.47	67.18	0.71
15	0.28	57.01	0.35
16	0.28	57.01	0.14
19	0.17	19.16	0.14
20	0.13	10.00	0.05
24	0.08	3.49	0.00
25	0.00	3.47	0.00
29	0.00	0.20	0.00

[a]For purposes of this study it was assumed that a subgradient requires $n+1$ experiments.

TABLE 2. *Difference between upper and lower bounds for problem 2*

(Optimal value = 785.8)

Number of experiments	(OLR)	(ILR)	(SIO)
4	5395.64	5397.61	5395.64
5	5257.87	5397.61	5257.87
9	5257.87	2362.39	5257.87
10	2.91	2335.85	2.91
14	0.18	334.58	0.18
15	0.04	231.49	0.04
16	0.00	231.49	0.00
⋮	⋮	⋮	⋮
44	0.00	0.55	0.00
45	0.00	0.00	0.00

TABLE 3. *Difference between upper and lower bounds for problem 3*

(Optimal value = 898.5)

Number of experiments	(OLR)	(ILR)	(SIO)
4	10.47	534.46	10.47
5	6.89	534.46	6.89
9	2.59	68.70	2.59
10	0.58	68.63	0.58
14	0.21	18.91	0.21
15	0.01	17.96	0.01
19	0.00	7.95	0.00
20	0.00	3.29	0.00
24	0.00	0.63	0.00
25	0.00	0.50	0.00
29	0.00	0.00	0.00

TABLE 4. *Difference between upper and lower bounds for problem 4*

(Optimal value = 13.6)

Number of experiments	(OLR)	(ILR)	(SIO)
2	5.96	22.91	5.96
3	5.96	22.91	5.96
5	4.47	11.51	4.47
6	4.47	11.51	4.47
8	2.61	5.73	2.61
9	2.61	5.73	2.61
11	1.48	2.86	1.48
12	1.48	2.86	1.48
14	0.86	1.63	0.86
15	0.52	1.61	0.52
17	0.18	0.91	0.18
18	0.05	0.52	0.11
20	0.03	0.17	0.05
21	0.01	0.16	0.03
23	0.01	0.06	0.01
24	0.00		0.01
26	0.00		0.01
27	0.00		0.00
29	0.00		0.00

TABLE 5. *Difference between upper and lower bounds for problem 1A*

(Optimal value = 219.2)

Number of experiments	(OLR)	(SIO)
4	637.41	637.41
5	637.41	467.36
6	637.41	340.93
9	478.40	340.93
10	171.93	30.88
11	171.93	11.16
12	171.93	9.19
14	18.60	9.19
15	8.04	9.19
16	8.04	1.49
17	8.04	0.51
19	3.68	0.51
20	2.09	0.51
21	2.09	0.09
24	0.94	
25	0.45	
29	0.11	
30	0.03	

that (ILR) is dominated by both (OLR) and (SIO). Comparison between (OLR) and (SIO) is more difficult. In Problems 2, 3, and 4 (OLR) and (SIO) yield identical results. In Problem 1 (SIO) out-performed (OLR) over most of the range.

With this information, it was conjectured that the relative advantage of (SIO) (with its inner-linearized, restricted subproblem at Step 2 as opposed to no subproblem at Step 2 for (OLR)) depends upon the degree of nonlinearity in the problem. That is, if f and $\mathbf{s}$ are highly nonlinear over the range of $\mathbf{x}$, (SIO) should have an advantage over (OLR) since the inner-linearized, restricted problem has

an opportunity to improve significantly the lower bound within the convex hull $\mathbf{x}^j$, $j \epsilon P$. Problem 1A was created to test this hypothesis. Results, shown in Table 5, tend to substantiate the hypothesis since the performance of (SIO) improves relative to (OLR).

The effectiveness of the natural termination criterion can be judged by comparing the value of the payoff function less cost of experimentation achieved using it with the value obtained using an ex post facto optimal rule. Results indicate that the cost range over which it is beneficial to undertake optimization with the natural termination criterion varies from approximately 0.56 to 0.87 of the cost range over which it is beneficial with an optimal stopping rule ([7] includes a more detailed discussion).

Solution times were short enough so that cost of auxiliary manipulations were small compared with the cost of experimentation over a realistic range of experimental costs.

6. ACKNOWLEDGEMENT

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APPENDIX

Problem 1 was formulated with the use of second-order polynomials and is specified below:

(A) $$\underset{\mathbf{x}}{\text{Maximize}}\ f_1(y_1)+f_2(y_2)-x_1y_3-x_2y_4-f_3(x_3)-f_4(x_4)-f_5(x_1)$$

subject to

1. $\mathbf{x} \geqslant 0$,
2. $x_1 \leqslant 150$,
3. $0.5x_1 \leqslant x_2 \leqslant 1.5x_1$,
4. $x_3 \leqslant 0.2x_1$,
5. $x_4 \leqslant 0.2x_1$,
6. $y_1 = s_{11}(\mathbf{x})$,
7. $y_2 = s_{12}(\mathbf{x})$,
8. $y_3 = s_{21}(x_1)$,
9. $y_4 = s_{22}(x_2)$,

where

$f_1(y_1) = 4225 = (65-0.1y_1)^2$,
$f_2(y_2) = 1156 - (34-0.2y_2)^2$,
$f_3(x_3) = 0.5(0.4x_3+2)^2-2$,
$f_4(x_4) = 0.5(-0.4x_4+2)^2-2$,
$f_5(x_1) = (0.2x_1-10)^2$,

and the actual functions used to represent s were

$s_{11}(\mathbf{x}) = 56.25-[7.5-0.015x_1-0.01x_2-0.012(x_3-x_4)]^2$,
$s_{12}(\mathbf{x}) = 20.25-[4.5-0.01x_1-0.008x_2-0.008(x_3-x_4)]^2$,
$s_{21}(x_1) = 1+(0.0125x_1)^2$,
$s_{22}(x_2) = 0.75+(0.01x_2)^2$.

The problem can be interpreted as an allocation problem for a firm in which the inputs are factors of production and the outputs are production quantities and factor prices. It was designed to be repre-

sentative, but small enough to allow solution by a variety of algorithms in a reasonable amount of computer time. The results for Problem 1 were obtained with initial conditions: $x_1 = 50.0$, $x_2 = 50.0$, $x_3 = 0.0$, $x_4 = 0.0$. Problem 1A is identical with the exception that initial conditions were:

$$x_1 = x_2 = x_3 = x_4 = 0.0.$$

Results for each algorithm are graphically depicted by the plot of upper and lower bounds in Figure A1 and by Table A1 which gives the sequence of solution points.

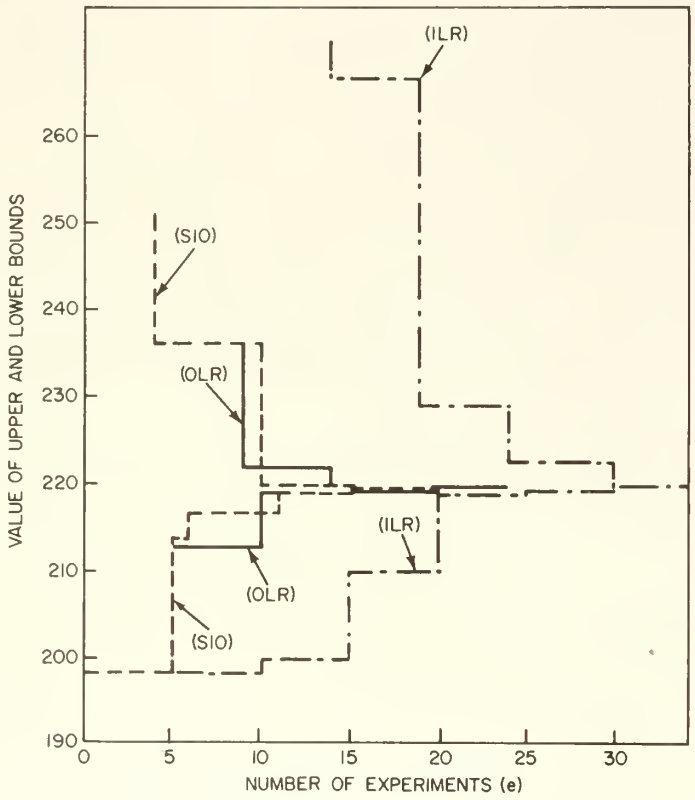

FIGURE A1. Value of upper and lower bounds vs number of experiments—Problem 1

TABLE A1. *Solution points for problem 1 using (OLR), (ILR), and (SIO)*

Number of experiments	x_1			x_2			x_3			x_4		
	(OLR)	(ILR)	(SIO)	(OLR)	(ILR)	(SIO)	(OLR)	(ILR)	(SIO)	(OLR)	(ILR)	(SIO)
0	50.00	50.00	50.00	50.00	50.00	50.00	0.00	0.00	0.00	0.00	0.00	0.00
4	62.31	150.00	62.31	80.63	225.00	80.63	7.33	30.00	7.33	0.00	0.00	0.00
5	.....	50.00	59.56	.....	50.00	73.81	.....	0.00	5.70	.....	0.00	0.00
6	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
9	56.43	100.00	.....	65.32	131.50	.....	9.73	20.00	.....	0.00	0.00	.....
10	.....	53.59	56.17	.....	56.18	65.27	.....	1.43	10.11	.....	0.00	0.00
11	.....	.....	56.49	.....	.....	66.07	.....	.....	9.69	.....	.....	0.00
14	56.88	71.55	.....	69.75	90.75	.....	10.31	14.31	.....	0.00	0.00	.....
15	.....	58.90	56.83	.....	96.38	68.87	.....	5.91	10.23	.....	0.00	0.00
16	.....	.....	56.83	.....	.....	68.87	.....	.....	10.23	.....	.....	0.00
19	56.88	57.33	.....	67.96	70.37	.....	9.86	11.47	.....	0.00	0.00	.....
20	.....	57.33	56.83	.....	70.37	68.87	.....	11.47	10.23	.....	0.00	0.00
24	.....	58.25	.....	.....	60.19	.....	.....	.....	.....	.....	.....	.....
25	56.88	57.39	.....	67.96	69.64	.....	9.86	0.00	.....	0.00	0.00	.....

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MINIMAL DISCONNECTING SETS IN DIRECTED MULTI-COMMODITY NETWORKS

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ABSTRACT

The problem of finding minimal disconnecting sets for multi-commodity directed networks may be solved using an arc-path formulation and Gomory's all-integer integer programming algorithm. However, the number of network constraints may be astronomical for even moderately sized networks. This paper develops a finite algorithm similar to Gomory's, but requiring no more than m rows in the tableau, where m is the number of arcs in the network.

PRELIMINARY REMARKS

A multi-commodity directed network, $G[N, \mathscr{A}]$, is a set N of nodes and a set $\mathscr{A}$ of arcs such that each arc in $\mathscr{A}$ has an ordered relation to a unique pair of distinct nodes in N. We rule out arcs which originate and terminate at the same node. Furthermore, there is, associated with G, a set of *source* nodes $\{s_i: i=1, 2, \ldots, r\}$, and a set of *sink* nodes $\{t_i: i=1, 2, \ldots, r\}$, where all s_i and t_i are in N and $s_i \neq t_j$ for all $i=j$. We now attach positive integer capacities to each arc.

The problem is to find a set of arcs in $\mathscr{A}$ with minimal combined capacity and whose removal from A prevents the formation of a path in G from any source to its respective sink. The removal of such a set of arcs is said to *disconnect* the network.

We let m,n and r denote the numbers of arcs, nodes and commodities, respectively.

This problem is related to interdicting networks. It has applications in communications and logistics, and may, as well, be applied in "reinforcing" networks at their weak points [2].

FORMULATION

In this paper the word *path* will be restricted to denote a set of arcs connecting source to a sink as follows:

DEFINITION: A path P in $G[N, \mathscr{A}]$ is a sequence of arcs related to a commodity k such that

$$P=\{(s_k, n_{j_1}), (n_{j_1}, n_{j_2}), \ldots, (n_{j_{p-1}}, n_{j_p}), (n_{j_p}, t_k)\},$$

where (s_k, t_k) is the source-sink pair for commodity k.

DEFINITION: A set D of arcs is called an r *commodity disconnecting set* if there is no path in $G[N, \mathscr{A}-D]$ from s_k to t_k for $k=1, 2, \ldots, r$.

DEFINITION: A set E of arcs is called a *commodity k cutset* if E partitions N into sets X and $\bar{X}$ such that $x\epsilon X$, $y\epsilon\bar{X}$ for every arc (x, y) in E, $s_k\epsilon X$, $t_k\epsilon\bar{X}$, $X\cup\bar{X}=N$, $X\cap\bar{X}=\phi$ (the empty set) and

$$c(E)=\sum_{a\epsilon E} c(a).$$

In a single-commodity network the minimal disconnecting set capacity and minimal cutset capacity are always equal. This fact leads to the useful "max-flow min-cut" Theorem [3] and some important algorithms. The equality does not exist in the multi-commodity case and except for special instances the following two Remarks [8] contain the most that can be said. If D_o is a minimal multi-commodity disconnecting set for G and if for $i=1, 2 \ldots, r$, D_i is a cut set, then

$$c(D_o) \leqslant c\left(\bigcup_{i=1}^{r} D_i\right).$$

If D_o is a minimal multi-commodity disconnecting set for G, then there exist single commodity disconnecting sets D_i, $i=1, 2, \ldots, r$, such that

$$c(D_o)=c\left(\bigcup_{i=1}^{r} D_i\right).$$

The arc-path formulation of the problem is based on the idea that every path in G must be cut.

DEFINITION: Matrix A is an *arc-path incidence matrix* if it contains a row for each path in G and a column for each arc in G and if column j contains a 1 in row i when arc j is in path i, and a zero otherwise.

Arc-path Formulation (P1)

Find $\mathbf{x}^*$ so that

$$\mathbf{cx}^*=\min\{\mathbf{cx}: A\mathbf{x}-I\mathbf{s}=\mathbf{e},\ \mathbf{s}\geqslant\mathbf{0},\ \mathbf{x}\text{ binary}\},$$

where A is the arc-path incidence matrix, I is an identity matrix and $\mathbf{e}$ is a vector of all ones. A component x_j of vector $\mathbf{x}$ can be interpreted as follows: $x_j=1$ implies arc a_j should be cut, $x_j=0$ implies arc a_j should not be cut. Under this interpretation, each constraint requires that some arc of the corresponding path be cut.

This problem is amenable to solution by Gomory's algorithm; however, the number of rows in A may be astronomical for even relatively small networks. Okada [7] has shown that an upper limit on the number of paths in a 1-commodity network is 2^{m-n+1}. Since each row of A corresponds to a particular path from some source to its corresponding sink, the number of rows of A may be as large as $r2^{m-n+1}$.

In addition to the *network constraints* in (P1) it is customary to append *auxiliary constraints* of the form $\mathbf{x}=\mathbf{x}^N$ which permits retention of all variables in the basis throughout the algorithm even though the nonbasic variables x_i^N change at each iteration.

The initial Beale's tableau* used in Gomory's approach is shown in Figure 1.

It is our intention to show that a Gomory type algorithm may be initiated with only the $-z$ *row and the network incidence relationships.* The algorithm can be viewed as a relaxation procedure for integer

*For a discussion of Beale's tableau and integer programing see T. C. Hu [5].

programing with one important difference. Here, we drop rows when slack variables immediately enter the basis (not just when the objective decreases).

Gomory's algorithm (*IG*) is a dual simplex algorithm which first selects an unsatisfied constraint, then selects a low capacity nonbasic variable to depart the nonbasic set, and finally pivots using a Gomory cut equation derived from the unsatisfied constraint to direct the column operations.

Our modified Gomory algorithm (*MG*) will also perform these steps. The unsatisfied constraint is found by seeking a path in the network. If such a path is found then the network is not disconnected and a low capacity arc is removed. This procedure is followed until there are no remaining paths. *MG* stores only those constraints which have been altered by pivoting.

The only data stored in *MG*, other than the network incidence relationships, is the $-z$ row and those auxiliary constraint rows altered by pivoting. *MG* will use no more than $m+1$ rows while *IG* may require as many as $r2^{m-n+1}+m+1$ rows.

MG follows, along with an example. The proof of optimality and finiteness is given after the example. Notice that the unsatisfied constraint found by the shortest path procedure must be "updated" before it is used to derive a Gomory cut. Also notice that the column choice is based on the lexicographic minimum in the partial tableau used and that this tableau grows as the algorithm progresses. Because of this *MG* will produce a different sequence of pivot steps from *IG*.

The columns in Figure 1 and in the partial tableaux of *MG*–II will be labeled α_j, $j=0, 1, \ldots, m$.

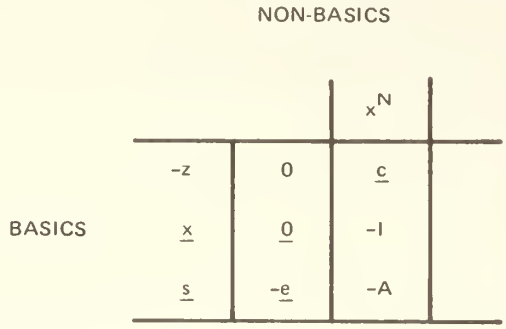

FIGURE 1. The full tableau

The pivot column index will be q and the pivot row temporarily appended to the bottom of the tableau will be denoted by $-a_o=-\mathbf{a}x^N+s$.

LEXICOGRAPHIC ORDERING

A vector $\mathbf{v}$ is lexicographically positive (l-pos) if it is not a zero vector and if its topmost nonzero component is positive (we write $\mathbf{v}>0$). Vector $\mathbf{v}$ is lexicographically greater (l–greater) than a vector $\mathbf{u}$ if $\mathbf{v}-\mathbf{u}>0$. Then lexicographical minimum (l-min) makes sense. However in the partial tableaux used in this paper some special provisions must be made.

The number of rows in the partial tableau may increase at each iteration and thus vectors with different dimensions must be compared lexicographically. The following rules are therefore useful.

1. If a vector is l-pos in its first p components, then it is l-pos.

2. If $\mathbf{v}$ is lexicographically minimum (l-min) in a set $\{\mathbf{v}_j\}$, then the vector formed from its first p components is l-min in the set $\{\mathbf{v}_j^!\}$ formed from the first p components of vectors in $\{\mathbf{v}_j\}$.

3. Convention: If $\mathbf{v}$ and $\mathbf{u}$ each have k components and if $\mathbf{w}=\begin{pmatrix}\mathbf{u}\\ \mathbf{x}\end{pmatrix}$ has $m(>k)$ components,

then we will say that $\mathbf{w}$ is an m-component *completion* of $\mathbf{u}$ and that $\mathbf{v} \succ \mathbf{w}$ whenever $\mathbf{v} \succ \mathbf{u}$.

Thus vectors of different dimension can sometimes be compared lexicographically. Notice that if $\mathbf{v} \succ \mathbf{u}$ then any m-component completion of $\mathbf{v}$ is l-greater than any m-component completion of $\mathbf{u}$, so the convention is logically consistent.

Modified Gomory Algorithm *MG*

STEP 0.

a. Construct the reduced tableau, Figure 2:

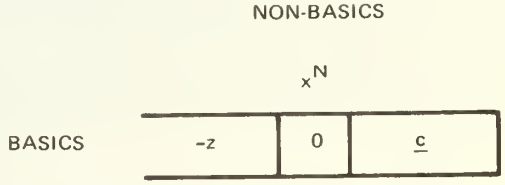

FIGURE 2. The initial *MG* tableau

b. Set $v=1$ and $D=\phi$.

STEP 1.

a. Select a commodity k.

b. Find a shortest path C from s_k to t_k in $G[N, \mathscr{A}-D]$.*

(1) If no path exists return to step 1.a.

(2) If no path exists for any commodity, terminate: D is a minimal disconnecting set.

c. Define path-vector constraint $\mathbf{c}_1$

$$(C1): -\mathbf{c}_1\mathbf{x}+s_v=-1,$$

where $c_{1j}=1$ if arc a_j is in path C and $c_{1j}=0$ otherwise.

STEP 2.

Define *modified* constraint $\mathbf{c}_2$ from $\mathbf{c}_1$

$$(C2): -\mathbf{c}_2\mathbf{x}^N+s_v=-1,$$

by adding to $(C1)$ each row from the tableau corresponding to a basic variable x_j for which $C_{1j}=1$.

STEP 3.

Select pivot column $\boldsymbol{\alpha}_q$ as an l-min column $\boldsymbol{\alpha}_j$ such that $c_{2j}<0$. Ties for l-min may be broken arbitrarily. If a column headed by an x_j^N is in the l-min set select it in preference to a column headed by an s_j^N.

STEP 4.

Define Gomory cut $(C3)$ using $(C2)$ as source row†

$$(C3)-\mathbf{c}_3\mathbf{x}^N+s_v=-1.$$

---

*Each arc may be assigned any nonnegative length, e.g., unity.

†The purpose of the Gomory cut is to ensure that the pivot element is minus one, thereby maintaining integrality of the new solution. Hu [5] presents a good description of Gomory cuts.

STEP 5.

a. Pivot: Set $\hat{\boldsymbol{\alpha}}_q = \boldsymbol{\alpha}_q$ and $\hat{\boldsymbol{\alpha}}_j = \boldsymbol{\alpha}_j + c_{3j}\boldsymbol{\alpha}_q$ for all $j \neq q$.

b. Relabel column $\boldsymbol{\alpha}_q$ with s_v^N and if the pivot column is headed by an x_j^N, then "add" the negative of (*C*3) to the bottom of the partial tableau. However, leave the pivot element equal to -1. If the pivot column is headed by an s_j^N, then discard row (*C*3).

STEP 6.

a. If $\boldsymbol{\alpha}_o$ contains a negative then set $v = v+1$ and select the topmost row with a negative in $\boldsymbol{\alpha}_o$ and label that row (*C*2). Return to step 3.

b. If $\boldsymbol{\alpha}_o$ is nonnegative then go to step 7.

STEP 7.

Define $D = \{a_i : x_i \geq 1\}$, set $v = v+1$ and return to step 1.

To illustrate this algorithm we will solve the Example given in Figure 3.

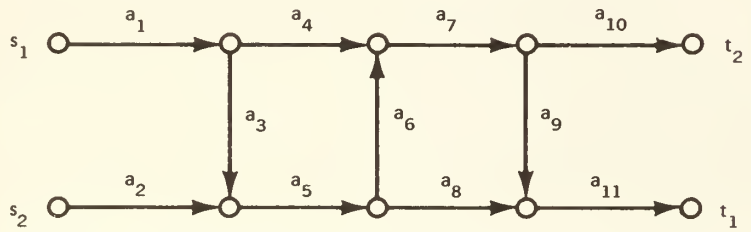

FIGURE 3. Example (All capacities are one)

Example 1: *MG* Solution

STEP 0. $v = 1$, $D = \phi$.

STEP 1. $k = 1$, $C = \{a_1, a_3, a_5, a_8, a_{11}\}$.

$$(C1): s_1 - x_1 - x_3 - x_5 - x_8 - x_{11} = -1.$$

STEP 2. Update

$$(C2): s_1 - x_1^N - x_3^N - x_5^N - x_8^N - x_{11}^N = -1.$$

STEP 3. $\boldsymbol{\alpha}_1$ is l-min.

STEP 4. Construct Gomory cut

$$(C3): s_1 - x_1^N - x_3^N - x_5^N - x_8^N - x_{11}^N = -1.$$

STEP 5. Pivot: See Figure 4(b).

STEP 7. $D = \{a_1\}$, $v = 2$.

STEP 1. $k = 2$, $C = \{a_2, a_5, a_6, a_7, a_{10}\}$*

$$(C1): s_2 - x_2 - x_5 - x_6 - x_7 - x_{10} = -1.$$

*For $k = 1$ there were no paths at all, i.e., no paths from source 1 to sink 1.

		x_1^N	x_2^N	x_3^N	x_4^N	x_5^N	x_6^N	x_7^N	x_8^N	x_9^N	x_{10}^N	x_{11}^N
-Z	0	1	1	1	1	1	1	1	1	1	1	1
(C_3): S_1	-1	-1*		-1		-1			-1			-1

(a)

		S_1^N	x_2^N	x_3^N	x_4^N	x_5^N	x_6^N	x_7^N	x_8^N	x_9^N	x_{10}^N	x_{11}^N
-Z	-1	1	1	0	1	0	1	1	0	1	1	0
X_1	1	-1		1		1			1			1
(C_3): S_2	-1		-1			-1*	-1	-1			-1	

(b)

		S_1^N	x_2^N	x_3^N	x_4^N	S_2^N	x_6^N	x_7^N	x_8^N	x_9^N	x_{10}^N	x_{11}^N
-Z	-1	1	1	0	1	0	1	1	0	1	1	0
X_1	0	-1	-1	1		1	-1	-1	1		-1	1
X_5	1		1			-1	1	1			1	
(C_3): S_3	-1	-1*	-1		-1		-1	-1		-1	-1	

(c)

		S_3^N	x_2^N	x_3^N	x_4^N	S_2^N	x_6^N	x_7^N	x_8^N	x_9^N	x_{10}^N	x_{11}^N
-Z	-2	1	0	0	0	0	0	0	0	0	0	0
X_1	1	-1		1	1	1			1	1		1
X_5	1		1			-1	1	1			1	

(d)

FIGURE 4. Example 1: *MG* tableaux (zero values have been omitted in other than the −Z row for clarity)

STEP 2. Update

$$(C2): s_2 - x_2^N - x_5^N - x_6^N - x_7^N - x_{10}^N = -1.$$

STEP 3. $\boldsymbol{\alpha}_5$ is l-min.

STEP 4. Construct Gomory cut

$$(C3): s_2 - x_2^N - x_5^N - x_6^N - x_7^N - x_{10}^N = -1.$$

STEP 5. Pivot: See Figure 4(c).

STEP 7. $D = \{a_5\}$, $v = 3$.

STEP 1. $k = 1$, $C = \{a_1, a_4, a_7, a_9, a_{11}\}$

$$(C1): s_3 - x_1 - x_4 - x_7 - x_9 - x_{11} = -1.$$

STEP 2. Update

$$(C2): s_3 - s_1^N - x_2^N + x_3^N - x_4^N + s_2^N - x_6^N - 2x_7^N + x_8^N - x_9^N - x_{10}^N = -1.$$

STEP 3. $\boldsymbol{\alpha}_1$ is l-min.

STEP 4. Construct Gomory cut

$$(C3): s_3 - s_1^N - x_2^N - x_4^N - x_6^N - x_7^N - x_9^N - x_{10}^N = -1.$$

STEP 5. Pivot: See Figure 4(d).

STEP 7. $D = \{a_1, a_5\}$, $v = 4$.

STEP 1. $k = 1$, $C = \phi$

$k = 2$, $C = \phi$, Terminate.

Solution: $D_o = \{a_1, a_5\}$, $c(D_o) = 2$.

PROOF OF FINITENESS FOR *MG*

The proof of finiteness follows the standard procedure of demonstrating lexicographic positiveness of columns in the partial tableau and then showing that the $\boldsymbol{\alpha}_o$ column is strictly lexicographically decreasing. It is necessary to show that considerations in the partial tableau are sufficient to continue with Gomory's proof [4: p. 200]. It will then remain to show that *MG* satisfies a proper row selection rule.

Problem (P1) contains only consistent constraints and has binary, feasible, and optimal solution.

THEOREM: After each pivot operation all columns $\boldsymbol{\alpha}_j$ in the partial tableau, except $\boldsymbol{\alpha}_0$, are lexicographically positive and all-integer.

PROOF: The initial tableau has $\boldsymbol{\alpha}_j > 0$ for $j = 1, \ldots, m$ since $\mathbf{c}$ is assumed to be positive and integer. We now show that if the theorem is true at iteration v, then it is true at iteration $v+1$ and the theorem follows inductively.

We emphasize that constraints $A\mathbf{x} - I\mathbf{s} = \mathbf{e}$ are implicit in the network and that (x, s) is the implicit set of basic variables. The partial tableau contains only a part of the solution. We will say that it contains an "explicit" or "partial" basis. Suppose that it contains $P+1$ rows, including the zero

row, at iteration v. Pivot column $\boldsymbol{\alpha}_q$ is always l-min in the set of columns with $c_{2j}<0$. The construction of the Gomory cut assures us that the new column,

$$\hat{\boldsymbol{\alpha}}_j=\boldsymbol{\alpha}_j+c_{3j}\boldsymbol{\alpha}_q,$$

will be lexicographically nonnegative through row P. Thus if the pivot row is "added" to the tableau all columns will be l-pos. Unfortunately the pivot row is not always "added" so we must consider cases.

In *IG* there is always a unique l-min pivot column by virtue of the linear independence of the auxiliary equations. In *MG* this cannot be used and, in fact, there may be many columns tied for l-min. It appears that pivoting will therefore produce columns of zeros in the partial tableau.

CASE 1: The variable heading the pivot column is an x_i^N. Suppose that pivot column $\boldsymbol{\alpha}_q$ is lexicographically equal to column $\boldsymbol{\alpha}_j$. Clearly $\hat{\boldsymbol{\alpha}}_j$ can be zero through row P only if $c_{3j}=-1$. But the negative of (*C*3) is appended to the tableau in this case so $\hat{\boldsymbol{\alpha}}_j$ is still l-pos.

CASE 2: The pivot column is headed by s_i^N. In this case only other slack variable columns can be in the l-min set since *MG* selects an x_i^N pivot column if one is available. In Case 2 the pivot row (*C*3) is not "added" to the tableau hence ties will produce zero column vectors. However, ties cannot occur among slack columns since these columns are originally linearly independent and the column operations performed on them cannot destroy this linear independence. Therefore in Case 2 the pivot column is uniquely l-min.

The partial tableau is initially all-integer. The pivot row is all-integer and the pivot element is -1 so the tableau remains all-integer.

THEOREM: Column vector $\boldsymbol{\alpha}_o$ lexicographically decreases by integer component changes in the partial tableau at every iteration.

PROOF: Pivot column $\boldsymbol{\alpha}_q$ is l-pos and pivot row (*C*3) contains a -1 in columns $\boldsymbol{\alpha}_o$ and $\boldsymbol{\alpha}_q$. Therefore $\hat{\boldsymbol{\alpha}}_o$ is equal to either $\begin{pmatrix}\boldsymbol{\alpha}_o-\boldsymbol{\alpha}_q\\1\end{pmatrix}$ or $\boldsymbol{\alpha}_o-\boldsymbol{\alpha}_q$ depending on whether or not (*C*3) is "added" to the tableau. In either case a strict integer valued lexicographical decrease occurs in the first $P+1$ components which implies the decrease in $\boldsymbol{\alpha}_o$.

THEOREM: Algorithm MG is finite.

PROOF: No more than m rows can appear in the partial tableau below the zero row because no variable can be introduced into the explicit basis twice and no slack can be introduced at all. More fundamentally, the partial tableau consists of a subset of the auxiliary constraints and there are just m of these. The auxiliary constraints not appearing in the partial tableau remain in their original form.

If we consider the network constraints as being implicitly listed below *all* auxiliary constraints, then the row selection rule in *MG* chooses the "topmost" row if an unsatisfied constraint appears in the partial tableau and chooses the shortest path in the subnetwork $G[N, \mathscr{A}-D]$ otherwise. The shortest path procedure does not necessarily yield a "topmost" row from the network constraints because it does not completely order these rows under all circumstances. An important unanswered question is: When does a shortest path procedure yield a complete ordering of the network constraints?

However, so long as there are unsatisfied constraints *MG* produces a pivot row and $\boldsymbol{\alpha}_o$ strictly l-decreases in the partial tableau and all columns remain l-pos (except $\boldsymbol{\alpha}_o$).

Because the topmost rule is invoked in the partial tableau, Gomory's finiteness proof may be used

there. That is, we know that the sequence of $\boldsymbol{\alpha}_0$ columns generated by pivoting is lexicographically bounded below by any solution and we know that there is a solution. Hence the topmost component of $\mathbf{a}_0$ cannot decrease indefinitely. In fact, it will reach a fixed value, say $-z^*$, and we may examine the remaining components. The second component cannot decrease indefinitely since if it were to do so it would become negative. It would then be selected as a source row and subsequently increased in value. But this is impossible once the first component is fixed. Thus the second component must reach a fixed value. Similarly all components of $\boldsymbol{\alpha}_0$ in the auxiliary rows must reach fixed values in a finite number of steps.

Now the network constraints must be satisfied also. For suppose some shortest path existed in the reduced network. Upon modifying this constraint and defining a Gomory cut we would have a pivot row with a -1 in $\mathbf{a}_0$ and a -1 pivot element. Since no element in the partial tableau $\mathbf{a}_0$ can change and implicit auxiliary rows cannot change either the pivot column must be all zero above the pivot row. This is a contradiction to the l-positiveness of all partial tableau columns. Therefore all constraints must be satisfied. This establishes the finiteness of *MG*.

PROOF OF OPTIMALITY FOR *MG*

MG does not generally produce the same sequence of pivot steps as *IG*. The shortest path procedure cannot be expected to yield a complete ordering of the network constraints. This is clear if the "lengths" used in the shortest path algorithm are changed as the algorithm progresses (as they may be). It is less clear when the "lengths" are not changed. If it were possible to construct the shortest path procedure so that it would consistently pick the same shortest path, say C, in $G[N, \mathscr{A} - D]$ for any D such that C was a shortest path, then a complete ordering would result. We do not have this assurance.

All of this notwithstanding *MG* uses elementary column operations to arrive at a dual-feasible, primal-feasible solution. Such a solution must be optimal.

COMPUTATIONAL RESULTS: *MG*

MG has been programmed in ALGOL and run on a Burroughs B–5500 computer. Problems were generated by a sub-program which randomly selected initial and terminal nodes where the number of nodes and the number of arcs is stipulated. Three-commodity problems with 300 arcs and 40 nodes were solved in approximately 12 min using about 35 iterations. Three-commodity, 300 arc, 30 node problems required about 6 min and 38 iterations.

Results for certain cases have been graphed for comparative purposes. The graphs (Fig. 5 and 6) show the effects of variations in the number of commodities upon the number of rows in the tableau, the number of iterations, and the process time. Next, a series of graphs shows the effect on the numbers of rows, iterations and on the process time of varying the number of arcs in a three-commodity network.

Each vertex in the graphs represents the average of nine or 10 problems. The maximum for the problem set is also shown on the graph with a dashed line.

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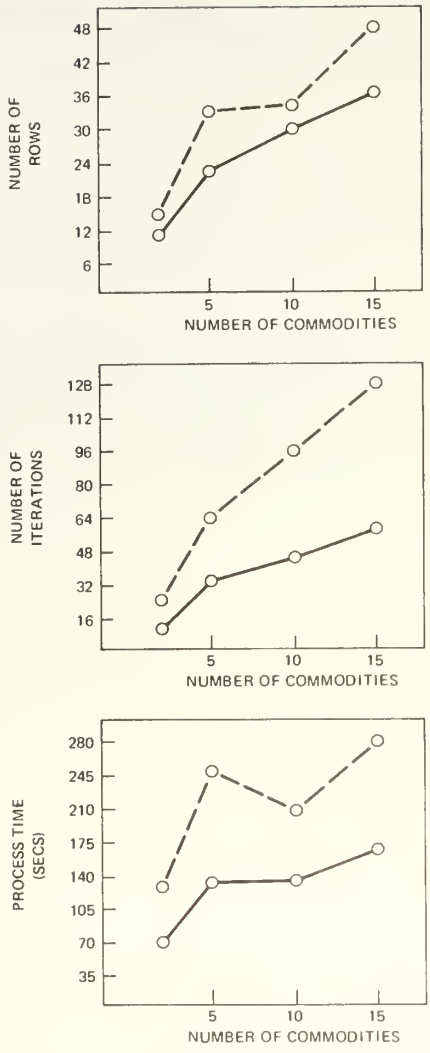

FIGURE 5. 100-Arc, 30-Node graphs

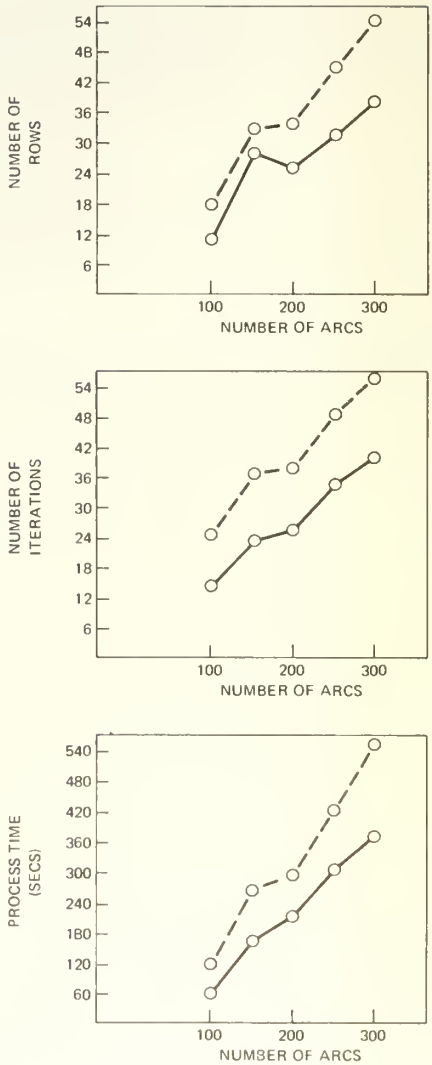

FIGURE 6. Three-commodity, 30-Node graphs

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DUALITY IN FRACTIONAL PROGRAMMING*

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ABSTRACT

This paper develops theoretical and computational aspects of the dual problem in linear fractional programming. This is done on the basis of two alternative methods for solving the primal fractional programming problem, both of which were presented in earlier literature. Parametric changes in the resource-vector are considered, and attention is given to infinitesimal as well as to discrete changes.

INTRODUCTION

Consider the fractional programming problem

$$\text{maximize } Q(x) = \frac{N(x)}{D(x)} = \frac{cx+\alpha}{dx+\beta}$$

subject to

(P1)
$$\begin{aligned} Ax &= b \\ x &\geqslant 0. \end{aligned} \qquad x \epsilon R^n$$

The first methods for solving this problem were apparently proposed by Dinkelbach [3] and Charnes and Cooper [2], of which the latter is the easiest to use. It applies the simplex-method to a simple transformation of the original problem (P1), and the problem can therefore be solved using a regular simplex-algorithm for linear programs.

Martos [10] suggested an algorithm for applying the simplex-method to the original problem (P1), but with modified entry criterion based on the gradient.† Swarup [12] describes an algorithm equivalent to Martos' and also suggests how to solve (P1) when the variables are required to be integers.

Some results connecting the solutions of linear fractional programming and those of parametric linear programming are obtained in Jagannathan [5] and Dinkelbach [4]. In Kydland [8] we describe an algorithm based on a parametric programming formulation of (P1). The article also contains an application of fractional programming to linear operations.

Some work has been done on formulating a dual problem to (P1) and calculating dual variables. Swarup considered theoretical aspects in [13], and went on to compute dual variables in [14] and [15].

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†Wagner and Yuan [16] have shown that this algorithm is algorithmically equivalent to Charnes and Cooper's method, in the sense that the variables of (P1) enter the basis in the same order.

However, he has a pair of dual variables for each constraint, one being dual with respect to the numerator and one for the denominator of the objective function. Chadda [1] in his decomposition algorithm implicitly computes the same pair of dual variables as Swarup does.

Kaska [6] found the correct formulation of the dual variables; however, he incorrectly gives the dual variables the interpretation of evaluating unit changes of the resource-vector b instead of infinitesimal changes. The reason for this difference from linear programming is, of course, that the objective function is nonlinear.

In the next section we will state assumptions and formulate a dual problem to (P1). Then a section is devoted to a brief description of Charnes and Cooper's solution method, and we explain how to compute the dual variables based on their method. A comparison with Martos' algorithm is offered, and here attention is also given to discrete changes of the resource vector b. The last section contains a numerical example.

PRELIMINARIES

Referring to (P1), let $S = \{x|Ax = b,\ x \geq 0\}$. We make the following assumptions

(a) $$S \neq \emptyset$$

(b) $$S \cap \{x|D(x) = 0\} = \emptyset$$

Without restricting the problem, we may also assume that $D(x) > 0$. If our original function is such that $D(x) < 0$, then we can instead maximize $Q(x) = \frac{-N(x)}{-D(x)}$, where $-D(x) > 0$.

According to Wolfe [17] the dual problem to (P1) is

(D1) $$\text{minimize } Q(x) - u(Ax-b)+vIx$$

subject to: $$\nabla Q(x) - uA + vI = 0$$

(1) $$v \geq 0.$$

In the following we will make use of the Kuhn-Tucker necessary optimality theorem for differentiable functions.* Since our constraints are all linear, (P1) clearly satisfies the constraint qualification. We therefore have:

THEOREM 1: Let $\bar{x}$ solve (P1). Then there exists $\bar{u} \in R^m$ and $\bar{v} \in R^n$ such that

(KT)
$$\nabla Q(\bar{x}) - \bar{u}A + \bar{v}I = 0$$

$$\bar{u}(A\bar{x}-b) - \bar{v}I\bar{x} = 0$$

$$\bar{v} \geq 0.$$

In view of Theorem 1 we see that at the optimum the value of the objective function of (D1) becomes

*See Kuhn and Tucker [7] or Mangasarian [9].

$$Q(\bar{x}) - \bar{u}(A\bar{x} - b) + \bar{v}I\bar{x} = Q(\bar{x});$$

i.e., the same as for (P1).

It may also be noted that since $A\bar{x} = b$, (KT) implies

$$x_j > 0 \Rightarrow \frac{\partial Q(\bar{x})}{\partial x_j} - \bar{u}a_j = 0.$$

In the following, we will show how the dual variables can be found using first the method proposed by Charnes and Cooper [2], and then the algorithm by Martos [10].

CHARNES AND COOPER'S METHOD

We make the transformation

$$y \equiv tx,$$

where $t = \dfrac{1}{dx + \beta}$. Then the primal problem (P1) can be written

(P2) maximize $L(y, t) \equiv cy + \alpha t$

subject to

(2) $$Ay - bt = 0$$

(3) $$dy + \beta t = 1$$

$$y \geq 0, \qquad t \geq 0.$$

The dual of (P2) is

(D2) minimize λ

subject to

(4) $$wA + \lambda d \geq c$$

$$-wb + \lambda B \geq \alpha.$$

From linear programming theory we know that if $(\bar{y}, \bar{t})$ is an optimal solution to (P2), then there exists $(\bar{w}, \bar{\lambda})$ optimizing (D2) and such that $\bar{\lambda} = L(\bar{y}, \bar{t}) = Q(\bar{x})$, where $\bar{x} = \bar{y}/\bar{t}$. This means that the dual variable corresponding to constraint (3) has the value of the maximand we are looking for. What is then the relationship between w in (D2) and u in (D1)? Let us write (1) more explicitly

$$\frac{1}{dx + \beta}\left(c - \frac{cx + \alpha}{dx + \beta}\, d\right) - uA + vI = 0,$$

i.e., $uA + \dfrac{cx + \alpha}{dx + \beta}\, td \geq tc$, or

(5) $$\frac{1}{t}uA+\frac{cx+\alpha}{dx+\beta}d\geqslant c.$$

At the optimum $\lambda=\dfrac{cx+\alpha}{dx+\beta}$, and so (5) is equivalent to (4) if $w=\dfrac{1}{t}u$, or $u=tw$.

CONCLUSION: Solving the problem (P2) will give us one dual variable corresponding to each constraint. The dual variables of (P1) are found by multiplying the dual variables for the constraints (2) by t. The dual variable for constraint (3) has the value at the optimum of the function $Q(x)$ that we are maximizing.

MARTOS' ALGORITHM

This algorithm is based on a direct application of the simplex-algorithm, with the exception that the entry criterion for changing the basis is different. The algorithm is well explained in Martos [10] and we will only report our results here.

Assume that the algorithm has been applied to the problem (P1). Then the following quantities can be found in the final tableau

$$\bar{c}^B=c^BB^{-1} \qquad \text{and} \qquad \bar{d}^B=d^BB^{-1},$$

where B is the final basis, and c^B and d^B denote the subvectors of c and d corresponding to vectors in the basis. $\bar{c}^B$ and $\bar{d}^B$ are similar to dual variables in linear programming, except that here we have two sets of them, one set corresponding to the numerator and one to the denominator.

By use of Theorem 1 it is easy to show Theorem 2.

THEOREM 2: If (P1) has a nondegenerate optimal solution $\bar{x}$, then

(6) $$\bar{u}=(\bar{c}^B-Q(\bar{x})\,\bar{d}^B)/D(\bar{x}).$$

Since the objective function is nonlinear, the u_i's will, in general, give the rate of change of the objective function only for infinitely small changes in the b_i's. An advantage with Martos' algorithm in terms of postoptimization is that one can easily calculate the change in Q for changes in b by any given amounts that do not result in a change of basis, or in general an upper bound for ΔQ. The formula when b changes to $b+h$ is

(7) $$\Delta Q=\frac{N+\bar{c}^Bh}{D+\bar{d}^Bh}-\frac{N}{D}=\frac{(\bar{c}^B-Q\bar{d}^B)h}{D+\bar{d}^Bh}=\frac{D\bar{u}h}{D+\bar{d}^Bh}.$$

When Charnes and Cooper's method is used, the change in b will show up as a change in the column corresponding to the variable, t, in the coefficient matrix for the LP-program, and we know that this column will always be in the basis. It is therefore clear that analysis of changes in b here becomes much more laborious.*

*See Simonnard [11], p. 157.

NUMERICAL EXAMPLE: Consider the example

$$\text{maximize } Q(x) = \frac{N(x)}{D(x)} = \frac{3x_1 + 5x_2 + 2x_3 - 1}{x_1 + 3x_2 + x_3 + 5}$$

subject to:

$$3x_1 + 2x_2 + x_3 + x_4 = 2$$

$$x_1 + 4x_2 + x_3 + x_5 = 1.$$

Following Charnes and Cooper, we let $t \equiv \dfrac{1}{x_1 + 3x_2 + x_3 + 5}$, and formulate the problem as follows

$$\text{maximize } 3y_1 + 5y_2 + 2y_3 - t,$$

subject to:

$$3y_1 + 2y_2 + y_3 - 2t + y_4 = 0$$

$$y_1 + 4y_2 + y_3 - t + y_5 = 0$$

$$y_1 + 3y_2 + y_3 + 5t = 1.$$

Applying the simplex-method, we obtain the following final tableau:

	c_j	3	6	2	−1	0	0	
c_j^B		y_1	y_2	y_3	t	y_4	y_5	
3	y_1	1	$-\frac{13}{12}$	0	0	$\frac{1}{2}$	$-\frac{7}{12}$	$\frac{1}{12}$
2	y_3	0	$\frac{59}{12}$	1	0	$-\frac{1}{2}$	$\frac{17}{12}$	$\frac{1}{12}$
−1	t	0	$-\frac{1}{6}$	0	1	0	$-\frac{1}{6}$	$\frac{1}{6}$
$z_j - c_j$		0	$\frac{21}{12}$	0	0	$\frac{1}{2}$	$\frac{5}{4}$	$\frac{1}{4}$

This tableau is reached after three iterations starting with basis variables y_4, y_5, and t. The solution to the original problem is

$$x_1 = \frac{\bar{y}_1}{\bar{t}} = \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2}$$

$$x_2 = \frac{\bar{y}_2}{\bar{t}} = 0$$

$$x_3 = \frac{\bar{y}_3}{\bar{t}} = \frac{1}{2} .$$

The dual variables above are*

$$\bar{w}_1 = \frac{1}{2}, \bar{w}_2 = \frac{5}{4}, \bar{\lambda} = \frac{1}{4} .$$

Therefore,

$$\bar{u}_1 = \bar{t}\bar{w}_1 = \frac{1}{12}$$

$$\bar{u}_2 = \bar{t}\bar{w}_2 = \frac{5}{24}$$

$$Q(\bar{x}) = \bar{\lambda} = \frac{1}{4} .$$

Martos' algorithm would, of course, have given the same solution, with dual variables calculated from (6). If, however, we want to know the change in Q if for instance b_2 increases by one unit, we can use formula (7), which gives the result $\Delta Q = \frac{5}{28} \neq \bar{u}_2$.

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EQUIVALENT KNAPSACK-TYPE FORMULATIONS OF BOUNDED INTEGER LINEAR PROGRAMS: AN ALTERNATIVE APPROACH

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ABSTRACT

In this paper we show that every bounded integer linear program can be transformed into an integer program involving one single linear constraint and upper and lower bounds on the variables, such that the solution space of the original problem coincides with that one of the equivalent knapsack-type problem.

INTRODUCTION

We consider the linear system of diophantine equations

$$\text{(P)} \qquad Ax = b$$

$$x_j \geq 0 \text{ and integer}, \qquad j \epsilon N = \{1, \ldots, n\}$$

where A is a $m \times n$ matrix of integers a_{ij}, $i \epsilon M = \{1, \ldots, m\}$, $j \epsilon N$ and b is a $m \times 1$ vector of integers. Throughout the paper we shall make the assumption that the space of feasible solutions to (P) is contained in a bounded subset K of R^n, i.e.,

$$\text{(0.1)} \qquad F(A, b) = \{x \epsilon R^n / Ax = b, \quad x \geqq 0, \, x \text{ integer}\} \subset K.$$

We can then assume without loss of generality that K is rectangular, i.e., K is of the form

$$\text{(0.2)} \qquad K = \{x \epsilon R^n / 0 \leq x_j \leq K_j, \quad j \epsilon N\},$$

where the K_j are upper bounds on the variables x_j, $j \epsilon N$.

The main result of this paper is the following theorem:

THEOREM: Given problem (P) and integers K_j, $j \epsilon N$, such that $x_j \leqq K_j$, $j \epsilon N$. There exists integers $q = (q_1, \ldots, q_m)$ such that –

$$\text{(0.3)} \qquad F(A, b) = F(qA, qb) \cap K,$$

where $F(A, b)$, $F(qA, qb)$, and K are defined as in (0.1) and (0.2).

The theorem states that every bounded integer linear program can be written equivalently as a

knapsack-type problem involving one single equation and the same number of variables as the original problem (in equality form (P)).

For the case of a matrix A with all positive coefficients a similar result has been obtained by Elmaghraby and Wig [5]. (I am indebted to Egon Balas, for pointing out to me the close relationship of Elmaghraby and Wig's paper and mine.) Bradley [3] and Glover and Woolsey [6] have generalized the original approach of Elmaghraby and Wig so as to accommodate for arbitrary coefficients in the matrix A (Bradley [3]) and so as to reduce the size of the multipliers (Glover and Woolsey [6], who continue to use the nonnegativity assumption of Elmaghraby and Wig). More recently, Anthonisse [1] has generalized the results of [8] with the aim to obtain multipliers of smaller magnitude.

Our work on the problem of reducing a set of diophantine equations into a single one differs from that of Elmaghraby et al. in a considerable aspect: Whereas Elmaghraby et al. base their results on the combination of two single diophantine equations—and use this idea successively to aggregate any set of m diophantine equations in problem (P), the problem is seen here as one of combining simultaneously m diophantine equations into a single one. More precisely, we define a mapping $y(x)$ of R^n on R^m as follows:

$$y(x) = Ax - b. \tag{0.4}$$

By the integrality of A and b, all integer points in R^n are mapped into integer points of R^m, in particular all feasible solutions to (P) are mapped into the origin of R^m. Of course, $y(x)$ is a many-to-one mapping which is generally not surjective. Furthermore, the image of any bounded subset $K \subseteq R^n$ under $y(x)$ is a bounded subset $L^* \subseteq R^m$. For ease of exposition, we shall consider the rectangular subset $L \supseteq L^*$ given by

$$L = \{y \in R^m / L_i \leq y_i \leq U_i,\ i = 1, \ldots, m\}, \tag{0.5}$$

where

$$L_i = \min_{x \in K} \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right) \tag{0.6}$$

$$i \in \{1, \ldots, m\}$$

$$U_i = \max_{x \in K} \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right).$$

The assertion of the above theorem is seen to be reduced to the question of the existence of a hyperplane in R^m which contains the origin of R^m and no other integer point of L^*. In the following section we show that there exists in fact a hyperplane in R^m the intersection of which with the integer points of L (instead of L^*) yields exactly the origin of R^m. Since $L \supseteq L^*$, this is sufficient to prove the assertion of the above theorem.

1. A LEMMA:

The general solution to a system of linear diophantine equations

(1.1) $$Ax = b$$

$$x \text{ integer},$$

where A is a $m \times n$ matrix of integers of rank $m - r$, $0 \leq r \leq m$, and b is a $m \times 1$ vector of integers can be written as follows

(1.2) $$x = x^* + Ft,$$

where x^* is a particular solution to (1.1) and F is fundamental for A (or phrased differently, a fundamental system of solutions to the homogeneous system $Ax = 0$, x integer). For a $n \times (n - m + r)$ matrix consisting of integer numbers to be fundamental for A, F must satisfy the following requirements:

(1.3a) $$\text{rank } (F) = n - m + r,$$

(1.3b) $$AF = 0,$$

(1.3c) $$\text{g.c.d. } (F) = \pm 1;$$

g.c.d. (F) stands for "greatest common divisor" of F and is defined to be the g.c.d. of the determinants of all $\binom{n}{n-m+r}$ $(n-m+r) \times (n-m+r)$ minors of F. The preceding results are all classical results in the theory of diophantine equations, see e.g., H.J.S. Smith [9].

Let U_i, L_i, $i \epsilon M$, be any collection of finite integers such that for every $i \epsilon M$ the following relation holds

(1.4) $$U_i \geq 0 \geq L_i, \qquad i \epsilon M;$$

i.e., U_i is a finite nonnegative integer number, and L_i is a finite nonpositive number. Let us assume that furthermore

(1.5) $$U_i^2 + L_i^2 > 0 \qquad \text{for at least one } i \epsilon M.$$

Then the following statement is true.

LEMMA 1: Given U_i, L_i, $i \epsilon M$, as specified in (1.4) and (1.5). There exist integer numbers $q = (q_1, \ldots, q_m)$ such that the system of diophantine (in-)equalities

(1.6a) $$\sum_{i=1}^{m} q_i y_i = 0,$$

(1.6b) $$y_i \text{ integer}, \qquad i \epsilon M, \text{ and}$$

(1.6c) $$U_i \geq y_i \geq L_i, \qquad i \epsilon M,$$

has the unique solution $y_i = 0$, $\forall\ i \epsilon M$.

PROOF: The proof is constructive in the sense, that we shall provide numbers q_i, $i \epsilon M$, which allow to arrive at the desired conclusion. Consider the diophantine equation (1.6a). Then for any choice of $q = (q_1, \ldots, q_m)$ (1.6a) is solvable in integers since $y = 0$ is always a particular solution. Hence we can apply the classical theory. In fact, it is possible to determine a matrix F which is fundamental for q, see H.J.S. Smith [9], art 9. Such an F with elements f_{ij} is given by

$$f_{ij} = \begin{cases} d_{i-1}/d_i & i = j+1, \quad j \epsilon \{1, \ldots, m-1\} \\ r_{ij} & i \leq j \leq m-1, \quad i \epsilon \{1, \ldots, m-1\} \\ 0 & \text{elsewhere,} \end{cases} \tag{1.7}$$

where the r_{ij} are defined by

$$r_{ij} = -\lambda_{i-1} \prod_{k=i}^{j-1} \mu_k (q_{j+1}/d_{j+1}). \tag{1.7a}$$

In formulas (1.7) and (1.7a)

$$d_i = \text{g.c.d.} \quad (q_1, \ldots, q_i), \quad i \epsilon \{1, \ldots, m\}$$

and the λ_j and μ_k are any solutions to the diophantine equations

$$\lambda_j q_{j+1} + \mu_j d_j = d_{j+1}, \quad j \epsilon \{1, \ldots, m-1\}, \tag{1.8}$$

with the usual conventions that

$$\lambda_0 = 1 \text{ and } \prod_{k=i}^{i-1} \mu_k = 1.$$

Now suppose we choose q_i as follows

$$q_i = Q^{m-i}, \quad i \epsilon \{1, \ldots, m\} \tag{1.9}$$

with a suitable integer $Q \geq 1$. Then (1.7) becomes

$$f_{ij} = \begin{cases} -1 & i = 1, \quad j \epsilon \{1, \ldots, m-1\} \\ Q & i = j+1, \quad j \epsilon \{1, \ldots, m-1\} \\ Q-1 & i \leq j \leq m-1, \quad i \epsilon \{2, \ldots, m-1\} \\ 0 & \text{elsewhere,} \end{cases} \tag{1.10}$$

with the choice of $\mu_j = 1$ and $\lambda_j = 1 - Q$ for all j in (1.8). This is possible since using (1.9), (1.8) is equivalent to

(1.8′) $$\lambda_j + \mu_j Q = 1, \qquad j \in \{1, \ldots, m-1\}.$$

Hence we can choose λ_j and μ_j independent of j as specified above. Using (1.2) with the particular solution $y^* = 0$ we have that every solution to (1.6a) and (1.6b) can be written as

(1.11) $$y_i = \sum_{j=1}^{m-1} f_{ij} t_j, \qquad i \in M,$$

or equivalently

(1.12) $$y_i = \begin{cases} -\sum_{j=1}^{m-1} t_j, & i = 1 \\ Q t_{i-1} + (Q-1) \sum_{j=i}^{m-1} t_j, & i = 2, \ldots, m, \end{cases}$$

with t_j, $j \in \{1, \ldots, m-1\}$, any integer number. Imposing (1.6c) on the y_i, we see that it is sufficient to choose Q as follows in order to obtain the desired conclusion of Lemma 1.1:

(1.13) $$Q = \max_{i=2, \ldots, m} (U_i + 1, -L_i + 1)$$

For, using the last two inequalities and the formula for y_m from (1.12) we have that

(1.14) $$U_m \geq Q t_{m-1} \geq L_m$$

which by (1.13) has the only integral solution $t_{m-1} = 0$, whence $y_m = 0$. Now use backward induction to obtain the conclusion. Q.E.D.

2. ALTERNATIVE CONSTRUCTION OF MULTIPLIERS

In this section we present alternative multipliers which we obtain by applying the general approach taken in the previous section to the special case where $m = 2$. Inspection of the proof of Lemma reveals that all we have to do in order to determine multipliers is to study the fundamental matrix F defined by (1.7). For the case of $m = 2$ this is still possible directly, for the cases $m \geqq 3$ this task becomes more difficult and it appears that stronger results are needed to obtain smaller multipliers (whenever they exist) than the ones constructed in the proof of Lemma 1 or provided by the method of this section (see Remark 2.3). In fact, the outcome of this section is that one can generally obtain smaller multipliers than the ones used in the proof of Lemma 1.

Let q_1 and q_2 be any integer numbers such that

(2.1) $$\text{g.c.d. } (q_1, q_2) = \pm 1,$$

and let furthermore U_i, L_i, $i = 1, 2$ be as above in (1.4) and (1.5). We then can state the following corollary to Lemma 1:

COROLLARY 1: Let q_i, U_i, L_i be as above and suppose that $q_1 > 0$. The system of diophantine (in-)equalities,

(2.2a) $$q_1 y_1 + q_2 y_2 = 0.$$

(2.2b) $$y_i \text{ integer}, \quad i = 1, 2.$$

(2.2c) $$L_i \leqq y_i \leqq U_i, \quad i = 1, 2,$$

has the unique solution $y_1 = y_2 = 0$ if and only if either (i) or (ii) holds:

(i) $$q_2 > 0 \text{ and } \min([-L_1/q_2], [U_2/q_1]) = 0 = \min([U_1/q_2], [-L_2/q_1])$$

(ii) $$q_2 < 0 \text{ and } \min([-U_1/q_2], [U_2/q_1]) = 0 = \min([L_1/q_2], [-L_2/q_1])$$

where $[r]$ denotes the largest integer less or equal to r.

PROOF: Suppose $q_2 = 0$; then (2.2a) cannot have $y_1 = y_2 = 0$ as unique solution. Hence we may assume for the proof in both directions that $q_1 \neq 0 \neq q_2$; more specifically, we can fix the sign of q_1 as done in the assumptions of the theorem. Consider now the diophantine equation (2.2a) and (2.2b). From the classical theory we have that every solution to (2.2a) and (2.2b) can be written as

(2.3) $$y_1 = q_2 t, \quad y_2 = -q_1 t;$$

where t is any integer. In fact, by assumption we have g.c.d. $(q_1, q_2) = \pm 1$ and hence from $q_1 \neq 0 \neq q_2$ that the column vector $(q_2, -q_1)^T$ is fundamental for (q_1, q_2). Furthermore, $y_1 = y_2 = 0$ if and only if $t = 0$. Using (2.2c) the first statement of the theorem is equivalent to the assertion that

(2.4a) $$U_1 \geqq q_2 t \geqq L_1,$$

(2.4b) $$U_2 \geqq -q_1 t \geqq L_2, \text{ and}$$

(2.4c) $$t \text{ integer},$$

has the unique solution $t = 0$. Now suppose $q_2 < 0$; then we have the equivalent relations:

(2.5a) $$L_1/q_2 \geqq t \geqq U_1/q_2,$$

(2.5b) $$-L_2/q_1 \geqq t \geqq -U_2/q_1,$$

or if $q_2 > 0$ we have instead of (2.5a)

(2.6) $$U_1/q_2 \geqq t \geqq L_1/q_2.$$

From the integrality of t, (2.4a) and (2.4b) have the unique solution $t=0$ if and only if either

(2.7) $$q_2<0 \text{ and } (L_1/q_2<1 \text{ or } -L_2/q_1<1) \text{ and } (U_1/q_2>-1 \text{ or } -U_2/q_1>-1)$$

or

(2.8) $$q_2>0 \text{ and } (U_1/q_2<1 \text{ or } -L_2/q_1<1) \text{ and } (L_1/q_2>-1 \text{ or } -U_2/q_1>-1).$$

Using the fact that for every noninteger real d we have $[d]+[-d]=-1$, we can write (2.7) equivalently as (*ii*) and (2.8) equivalently as (*i*). Q.E.D.

REMARK 2.1: Using (*i*) of Corollary 1, it is sufficient to choose any two relative prime integers q_1 and q_2, such that one of the following relations holds:

(2.9a) $$q_1 \geqq U_2+1 \quad \text{and} \quad q_2 \geqq U_1+1,$$

(2.9b) $$q_1 \geqq -L_2+1 \quad \text{and} \quad q_2 \geqq -L_1+1,$$

(2.9c) $$q_1>0, \text{ arbitrary} \quad \text{and} \quad q_2 \geqq \max(U_1+1, -L_1+1), \text{ and}$$

(2.9d) $$q_1 \geqq \max\ (U_2+1, -L_2+1) \quad \text{and} \quad q_2>0, \text{ arbitrary.}$$

In order to satisfy (ii) it is sufficient to choose $q_2<0$ such that (2.9a)–(2.9d) is satisfied with q_2 replaced by $(-q_2)$. Of course, it is always possible to find two relative prime integers such that anyone of the relations (2.9a)–(2.9d) is fulfilled. This follows from the well known fact that there exist infinitely many prime numbers.

REMARK 2.2: From Remark 2.1 it is clear that not all of the integers U_i, L_i $i=1, 2$ need to be finite in (2.2c) for to obtain the desired conclusion that (2.2a)–(2.2c) has the unique solution $y_1=y_2=0$.

REMARK 2.3: From Corollary 1 we have an alternative proof of the theorem by first aggregating the first two constraints of the problem into one; then this new constraint and the originally third one into one, and so forth. Clearly, by always choosing the multipliers according to (2.9a)–(2.9d) such that $q_1^2+q_2^2$ becomes the smallest possible we will end up with smaller coefficients than the ones used to prove the theorem directly. But it is clear that these multipliers, too, become extremely large for a large m.

REMARK 2.4: It should be noted that Corollary 1 cannot provide a necessary condition on multipliers $q=(q_1, q_2)$ with the property that $F(A, b)=F(qA, qb)$, where $m=2$. This stems from the fact that the y_i are defined by a relation of the form (0.4), i.e.,

(2.10) $$y_i=y_i(x)=\sum_{j=1}^{n} a_{ij}x_j-b_i, \qquad i\epsilon M, x\epsilon K.$$

Hence, due to the integrality of the x as well as the coefficients, the image of K under the mapping (2.10) is generally a proper subset of

$$L = \{y \epsilon R^m / U_i \geqq y_i \geqq L_i,\ y_i \text{ integer}, i \epsilon M\}.$$

Consequently there may exist multipliers $q=(q_1, q_2)$ such that $F(A, b)=F(qA, qb)$ which do not satisfy neither (i) nor (ii) of Corollary 1. For example, consider the problem.

$$5x_1 + 5x_2 + 4x_3 = 9$$

$$2x_1 + 4x_2 + 3x_3 = 7$$

$$x_i = 0 \text{ or } 1, \qquad i = 1, 2, 3,$$

Then $q_1 = 1$, $q_2 = -1$ yields $F(A, b) = F(qA, qb)$ whereas the application of Corollary 1 (Remark 2.1) yields as multipliers with the smallest absolute sum $q_1 = 8$ and $q_2 = 1$ which coincide with the smallest multipliers obtained from Mathews' method (see Elmaghraby and Wig (5)). Of course, this remark applies analogously to the general approach taken in this paper.

3. A NUMERICAL EXAMPLE

In this section we discuss by means of an example the more obvious possibilities of how to use the preceding results in solving integer programming problems in zero–one variables. Unnecessary to say, the following is not and is not intended to be in any sense an exhaustive treatment, but it indicates that the preceding may be useful for the solution of integer linear problems.

We consider the following zero–one program

$$\min 5x_1 + 7x_2 + 10x_3 + 3x_4 + x_5,$$

subject to:

$$-x_1 + 3x_2 - 5x_3 - x_4 + 4x_5 + x_6 = -2$$

$$2x_1 - 6x_2 + 3x_3 + 2x_4 - 2x_5 + x_7 = 0$$

$$x_i = 0 \text{ or } 1, \qquad i = 1, \ldots, 5$$

$$x_6, x_7 \geqq 0, \text{ integer.}$$

It is easy to derive the following upper bounds on the slacks

(3.1) $$0 \leqq x_6 \leqq 5, \qquad 0 \leqq x_7 \leqq 8$$

and the following upper and lower bounds on $y_i(x)$ for $i = 1, 2$ (for the definition of $y_i(x)$ see (2.10)):

(3.2) $$-5 \leqq y_1(x) \leqq 14, \qquad -8 \leqq y_2(x) \leqq 15.$$

Using (2.9b) we can choose $q_1=9$, $q_2=7$ and we obtain after multiplication and addition the following single equation:

$$5x_1-15x_2-24x_3+5x_4+22x_5+9x_6+7x_7=-18.$$

Substituting $x_j'=1-x_j$ for j=2 and 3, we have that in every integer solution we must have $x_3=1$ and $x_5=0$ and furthermore that the slacks have sharper upper bounds:

(3.3) $$0\leqq x_6\leqq 2, \qquad 0\leqq x_7\leqq 3.$$

We can use this to derive a new set of smaller multipliers since now

(3.4) $$-5\leqq y_1(x)\leqq 2, \qquad -3\leqq y_2(x)\leqq 10,$$

for all solutions x with $x_3=1$ and $x_5=0$. Applying (2.9c) with $q_1=1$ and $q_2=6$ the resulting new equation after combination is obtained to be

$$11x_1-33x_2+11x_4+x_6+6x_7=-15$$

Again applying $x_2'=1-x_2$ shows that $x_2=1$ in every feasible solution, so that finally the problem becomes

$$\min 5x_1+3x_4+17$$

subject to:

$$11x_1+11x_4+x_6+6x_7=18$$

$$x_1,\ x_4=0 \text{ or } 1$$

$$0\leqq x_6\leqq 2$$

$$0\leqq x_7\leqq 3 \text{ integer.}$$

This problem has the optimal solution $x_1=x_4=x_6=0$ and $x_7=3$.

Of course, it will not always be possible to reduce the original problem as far as done here. Observe, however, that neither the first equation nor the second equation of the original problem imply that $x_2=x_3=1$ and $x_5=0$. After proper combination of the two equations, however, this becomes apparent. Hence, more generally, the method of combining several constraints into a single one may be used in connection with determining whether or not some facets (or if more refined, some faces) of the unit-cube are cut off by a collection of constraints (see [2] for a different approach to the same problem). The method of doing so is easy and requires nothing but additions, multiplications, and comparisons. Furthermore, it will frequently be possible to determine sharper bounds on the nonzero–one variables

which in turn has a decreasing effect upon the size of the multipliers (and hence on the size of the coefficients in the final knapsack-constraint). One way of using the general approach described here would thus consist of a "precomputational analysis," which would in general allow to reduce the number of variables involved.

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CONSTRUCTING BIMATRIX GAMES WITH SPECIAL PROPERTIES

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ABSTRACT

This work is concerned primarily with presenting techniques for constructing bimatrix games with predetermined equilibrium points. Some partial results on Nash-solvability are also given.

I. INTRODUCTION

A bimatrix game is defined by a pair (A, B) of real $m \times n$ matrices together with the Cartesian product $X \times Y$ of all m-dimensional probability vectors X and all n-dimensional probability vectors Y. A point $(\bar{x}, \bar{y})$ in $X \times Y$ is an equilibrium point of the game (A, B) if $\bar{x}A\bar{y}^t \geq xA\bar{y}^t$ for all $x \epsilon X$ and $\bar{x}B\bar{y}^t \geq \bar{x}By^t$ for all $y \epsilon Y$.

Previous work on such games has included the following: Kuhn [6] defined an extreme equilibrium strategy in a particular way and showed that all extreme equilibrium strategies can be obtained by examining square submatrices of the payoff matrices. Mills [10] showed that a pair (x, y) of mixed strategies is an equilibrium point if, and only if, there exist scalars α, β such that (x, y, α, β) solves: maximize $xAy + xBy - \alpha - \beta$

subject to

$$Ay \leq \alpha l,\ xB \leq \beta l,\ x \geq \phi_m,\ y \geq \phi_n,\ \sum_{i=1}^{n} x_i = \sum_{j=1}^{n} y_j = 1,$$

where l is a row or column vector each of whose entries is 1. Mangasarian [11] considered the same problem, and pointed to the use of the algorithm of Balinski [1] for finding all vertices of a polyhedral convex set in solving it. Lemke and Howson [7] showed, by an algebraic argument, that an equilibrium point lies on a path joining a sequence of adjacent extreme points of a certain convex polyhedron, and gave an algorithm which terminates either in an equilibrium point or in an unbounded edge of the polyhedral set. Raghavan [12] studied the properties of equilibrium points in completely mixed games.

The algebraic properties of 0-sum matrix games were explored thoroughly by Gale and Sherman [3] and by Bohnenblust, Karlin, and Shapley [2]; some of these properties were shown in [9] to generalize to a class of bimatrix games. Others not yet treated for the general bimatrix case include construction of a game with given equilibrium points. There is also the open question of characterizing those bimatrix games that are "Nash-solvable"—games whose equilibrium points are interchangeable (see [8], p. 106)—and in characterizing the solution of a Nash-solvable game (its set of equilibrium points). This work is concerned mainly with techniques for constructing bimatrix games having desired equi-

librium points, and has some partial results on Nash solvability. Other results motivated by the structures used are also included.

In the following, the inner product of two vectors a, b in E^n will be denoted by (a, b). Also assume $\alpha > 0, \beta > 0$.

II. CONSTRUCTING A GAME WITH GIVEN EQUILIBRIUM POINTS

THEOREM 1: Let $(\bar{x}, \bar{y})$ represent a pair of probability vectors in $X \times Y$ and let α, β be real numbers. There exists a game (A, B) having $(\bar{x}, \bar{y})$ as an equilibrium point and having α, β as the corresponding payoffs to P_1 and P_2, respectively.

PROOF: Assume without loss of generality that $\alpha > 0, \beta > 0$. Regard $\bar{x}$ and $\bar{y}^T$ as linear transformations from E^m and E^n to E^1, respectively. Let $b_1, b_2, \ldots, b_{m-1}$ and $a_1^T, a_2^T, \ldots, a_{n-1}^T$ be bases, respectively, for the nullspaces of $\bar{x}$ and $\bar{y}^T$. Let $a^T \epsilon E^n$ be such that $(a, \bar{y}) = \alpha$, let $b \epsilon E^m$ be such that $(\bar{x}, b) = \beta$. Let $M = \{1, 2, \ldots m\}$, $N = \{1, 2, \ldots n\}$. Let $M_1(\bar{x}) = \{i | \bar{x}_i > 0\}$, let $N_1(\bar{y}) = \{j | \bar{y}_j > 0\}$. Let $A_{i\cdot}$ be the ith row of A, $B_{\cdot j}$ the jth column of B. For $i \epsilon M_1(\bar{x})$, let $A_{i\cdot} = a + \sum_{k=1}^{n-1} \lambda_{ik} a_k$, λ_{ik} arbitrary real numbers, and for $j \epsilon N_1(\bar{y})$, let $B_{\cdot j} = b + \sum_{k=1}^{m-1} \gamma_{jk} b_k$. For $i \epsilon M - M_1(\bar{x})$, let $A_{i\cdot}$ be any vector in E^n such that $(A_{i\cdot}, \bar{y}) \leqslant \alpha$; $A_{i\cdot} = \frac{1}{2} a$ will do. For $j \epsilon N - N_1(\bar{y})$, let $B_{\cdot j}$ in E^m be such that $(\bar{x}, B_{\cdot j}) \leqslant \beta$; $B_{\cdot j} = \frac{1}{2} b$ will do. Then the game (A, B) has as an equilibrium point $(\bar{x}, \bar{y})$, since $\bar{x}_i = 0$ if $(A_{i\cdot}, \bar{y}) < \alpha$ and $\bar{y}_j = 0$ if $(\bar{x}, B_{\cdot j}) < \beta$, and $\max_i (A_{i\cdot}, \bar{y}) = \alpha$, $\max_j (\bar{x}, B_{\cdot j}) = \beta$.

It is, conversely, easy to show that any game (A, B) having $(\bar{x}, \bar{y}, \alpha, \beta)$ as an equilibrium point must have such a construction. Let $a_1^T, \ldots a_{n-1}^T$ be a basis for the nullspace of $\bar{y}^T$, let $(a, \bar{y}) = \alpha$. Then for $i \epsilon M_1(\bar{x})$, $A_{i\cdot}$ has a representation $\lambda_i a + \sum_{k=1}^{n-1} \lambda_{ik} a_k$, since $a_1, \ldots a_{n-1}, a$, are linearly independent and therefore a basis for E^n. Since by assumption $(A_{i\cdot}, \bar{y}) = \alpha = \lambda_i \alpha$, $\lambda_i = 1$, and the proof follows. Clearly, the set of all bimatrix games having $(\bar{x}, \bar{y}, \alpha, \beta)$ as an equilibrium point is characterized by this result.

An interesting problem is posed by the desire to construct a game (A, B) having $(\bar{x}, \bar{y})$ as its *unique* equilibrium point. This work will not resolve this question and it must for the moment remain open. A partial result can, however, be obtained.

DEFINITION 1: An equilibrium point $(\bar{x}, \bar{y})$ is said to be completely mixed if $\bar{x}_i > 0$, $i = 1 \ldots m$, and $\bar{y}_j > 0$, $j = 1, \ldots n$. The reader is at this point referred to Raghavan [12] for his work on completely-mixed strategies in bimatrix games.

THEOREM 2: Let $(\bar{x}, \bar{y})$ be a completely-mixed strategy pair, $\bar{x} \epsilon E^m$, $\bar{y} \epsilon E^n$. A necessary and sufficient condition for the existence of a game (A, B) that has $(\bar{x}, \bar{y})$ as its unique completely-mixed equilibrium point is that $m = n$.

PROOF: Assume $m = n$. Choose, as before, a basis $a_1^T, a_2^T \ldots a_{n-1}^T$ for the nullspace of $\bar{y}^T$, and let $a \epsilon E^n$ be such that $(a, \bar{y}) = \alpha$. Let $b_1, b_2, \ldots b_{n-1}$ be a basis for the nullspace of $\bar{x}$, and let $b \epsilon E^n$ be such that $(x, b) = \beta$. Let A be the $n \times n$ matrix whose ith row $A_{i\cdot} = a + a_i$, $i = 1, \ldots n-1$. Let $A_n = a$. Let B be the $n \times n$ matrix whose jth column $B_{\cdot j} = b + b_j$, $j = 1, \ldots n-1$; let $B_{\cdot j} = b$. Then A and B are nonsingular and $Ay = (\alpha \ldots \alpha)$, $xB = (\beta, \ldots \beta)$ are solved uniquely by $\bar{y}, \bar{x}$, respectively. Let (x^*, y^*) be another completely-mixed equilibrium point with payoffs α^*, β^* Then $Ay^* = (\alpha^* \ldots \alpha^*)^T$, $x^*B = (\beta^*, \ldots \beta^*)$, from which y^* is a scalar multiple of $\bar{y}$ and x^* is a scalar multiple of $\bar{x}$. It then

follows, from the fact that these are all probability vectors, that $x^*=\bar{x}$, $y^*=\bar{y}$, and therefore $\alpha^*=\alpha$, $\beta^*=\beta$.

To show necessity, suppose $m>n$. Let B be chosen to satisfy $\bar{x}B=(\beta \ . \ . \ . \ \beta)$. Since the rank of B is at most n, there is a solution $x^0\neq\phi$ to $xB=(0, \ . \ . \ . \ 0)$, no matter how B is chosen. Since $\bar{x}$ is completely mixed, there is a scalar $\lambda\neq 0$ such that $x'=\bar{x}+\lambda x^0$ has all positive components. If x' is scaled to sum to 1, $(x', \bar{y})$ is also a completely mixed equilibrium point.

Raghavan [12] showed (his Theorem 2) that if $m>n$, there is an x^*, not completely mixed, such that $(x^*, \bar{Y})$ is an equilibrium point. This also follows from the above by an appropriate choice of λ, when $\bar{y}$ is completely mixed, as it is here by assumption.

EXAMPLE: Let $\bar{x}=(\frac{1}{2}, \frac{1}{2})=\bar{y}^T$. Let $\alpha=1$, $\beta=\frac{1}{2}$. Let $a_1=(1, -1)$, $a=(1, 1)$, so that $(a, \bar{y})=1$, $(a_1, \bar{y})=0$. Let $A=\begin{pmatrix}1 & 1\\ 2 & 0\end{pmatrix}$. Let $b_1=(1, -1)^T$, $b=(1, 1)^T$, and $B=\begin{pmatrix}2 & 1\\ 0 & 1\end{pmatrix}$. Then $(\bar{x}, \bar{y})$ is the unique equilibrium point for (A, B). On the other hand, let $A'=\begin{pmatrix}2 & 0\\ 1 & 1\end{pmatrix}$, $B'=B=\begin{pmatrix}2 & 1\\ 0 & 1\end{pmatrix}$. Then $((1, 0), (1, 0)^T)$ and $((0, 1), (0, 1)^T)$ are additional equilibrium points for the game (A', B), even though the construction of the two games within the process described is identical. Thus the structure described is of no use in attempting to construct a game having the pair $(\bar{x}, \bar{y})$ as its unique equilibrium point. The following observation can, however, be asserted.

COROLLARY: Let $\bar{x}\epsilon E^n$, $\bar{y}\epsilon E^n$ be completely mixed. If (A, B) is constructed as described above and (x^*, y^*) is an equilibrium point of (A, B) other than $(\bar{x}, \bar{y})$ then neither x^* nor y^* is completely mixed.

The following results show other properties that games can be designed to have.

THEOREM 3: Let $(x^1, y^1, \alpha_1, \beta_1)$, $(x^2, y^2, \alpha_2, \beta_2)$ be given, $x^1\neq x^2$, $y^1\neq y^2$. Then there exists a game having (x^1, y^1) as an equilibrium point with payoffs α_1, β_1, and likewise having (x^2, y^2) as an equilibrium point with payoffs α_2, β_2, such that (x^1, y^2) and (x^2, y^1) are also equilibrium points. If $M_1(x^1)\neq M_1(x^2)$ or $N_1(y^1)\neq N_1(Y^2)$, there further exists a game with (x^1, y^1) and (x^2, y^2) as equilibrium points that does not have this interchangeability property.

PROOF: Let $M_2(y)=\{i \mid (A_i\cdot, y)=\max_k (A_k\cdot, y)\}$, $N_2(x)=\{j \mid (x, B\cdot_j)=\max_k (x, B\cdot_k)\}$. Let $x=\begin{pmatrix}x^1\\ x^2\end{pmatrix}$ be a $2\times m$ matrix. and let $b_1, b_2, \ . \ . \ . \ b_{m-2}$ be a basis for the nullspace of x regarded as a linear transformation from E^m to E^2 (x^1 and x^2 are obviously linearly independent, since $x^1\neq x^2$ and both are probability vectors). Let b^1 be in the nullspace of x^1 and not in nullspace of x^2, let b^2 be in nullspace of x^2 and not in nullspace of x^1. Assume with no loss of generality that $(x^2, b^1)=\beta_2$, that $(x^1, b^2)=\beta_1$. For $j\epsilon N_1(y^1)\cup N_1(y^2)$, let

$$B\cdot_j=b^1+b^2+\sum_{r=1}^{m-2}\lambda_{jr}b_r;$$

for $j\epsilon N-(N_1(y^1)\cup N_1(y^2))$, let $B\cdot_j$ satisfy $(x^1, B\cdot_j)\leq\beta_1$, $(x^2, B\cdot_j)\leq\beta_2$. Then for $j\epsilon N_1(y^1)\cup N_1(y^2)$, $(x^1, B\cdot_j)=\beta_1$, $(x^2, B\cdot_j)=\beta_2$, while otherwise $(x^1, B\cdot_j)\leq\beta_1$, $(x^2, B\cdot_j)\leq\beta_2$.

After a similar construction of A, it is clear that $x_i^1=0$ if $(A_i\cdot, y^1)<\alpha_1$, $y_j^1=0$ if $(x^1, B\cdot_j)<\beta_1$, and (x^1, y^1) is an equilibrium point for (A, B) with payoffs α_1, β_1, respectively, and likewise for $(x^2, y^2, \alpha_2, \beta_2)$. The equilibrium points are interchangeable.

It is clear that, for equilibrium points (x^1, y^1) and (x^2, y^2) to lack interchangeability, one must have either $M_1(x^1) \neq M_1(x^2)$ or $N_1(y^1) \neq N_1(y^2)$. Suppose $N_1(y^1) \neq N_1(y^2)$. Let b^1, b^2, and $b_1, \ldots b_{m-2}$ be as before.

For $j \epsilon N_1(y^1) \cap N_1(y^2)$, let

$$B \cdot_j = b^1 + b^2 + \sum_{k=1}^{m-2} \lambda_{kj} b_k.$$

If $j \epsilon N_1(y^1) - N_1(y^2)$, let

$$B \cdot_j = b^3 + \sum_{k-1}^{m-2} \lambda_{kj} b_k,$$

where $(x^1, b^3) = \beta_1 > 0$ and $(x^2, b^3) = 0$ (or at any rate, $(x^2, b^3) < \beta_2$. Such a b^3 obviously exists. If $j \epsilon N_1(y^2) - N_1(y^1)$, let

$$B \cdot_j = b^4 + \sum_{k=1}^{m-2} \lambda_{kj} b_k,$$

where $(x^2, b^4) = \beta_2 > 0$, $(x^1, b^4) < \beta_1$. For $j \notin N_1(y^1)$, $j \notin N_1(y^2)$, choose $B \cdot_j$ such that $(x^1, B \cdot_j) \leq \beta_1$, $(x^2, B \cdot_j) \leq \beta_2$. Again, such $B \cdot_j$ obviously exist. Then, clearly, $(x^1, B \cdot_j) = \beta_1$, $j \epsilon N_1(y^1)$, $(x^1, B \cdot_j) < \beta_1$, $j \epsilon N_1(y^2)$, $j \notin N_1(y^1)$; $(x^1, B \cdot_j) \leq \beta_1$, $j \notin N_1(y^1) \cup N_1(y^2)$; $(x^2, B \cdot_j) = \beta_2$, $j \epsilon N_2(y^2)$, $(x^2, B \cdot_j) < \beta_2$, $j \epsilon N_1(y^1)$, $j \notin N_1(y^2)$, and $(x^2, B \cdot_j) \leq \beta_2$, $j \notin N_1(y^1) \cup N_1(y^2)$. It is then clear that either y^2 is not in equilibrium with x^1 or y^1 is not in equilibrium with x^2 or both, and thus interchangeability does not exist with this construction. A similar construction can, of course, be made if $M_1(x^1) \neq M_1(x^2)$.

III. A PARTIAL CHARACTERIZATION OF NASH-SOLVABLE GAMES AND THEIR SOLUTIONS

The foregoing discussion provides some insight into the following question: Under what conditions are two equilibrium points interchangeable? In the 0-sum case, all equilibrium points are; the next theorem considers the more general case.

THEOREM 4: Let $(x^1, y^1, \alpha_1, \beta_1)$, $(x^2, y^2, \alpha_2, \beta_2)$ be two equilibrium points and payoffs, for a game (A, B). The necessary and sufficient condition for $(x^1, y^1, \alpha_1, \beta_1)$ and $(x^2, y^2, \alpha_2, \beta_2)$ to be interchangeable is for $M_1(x^1) \subset M_2(y^2)$, $M_1(x^2) \subset M_2(y^1)$, $N_1(y^1) \subset N_2(x^2)$, $N_1(y^2) \subset N_2(x^1)$.

PROOF: Suppose (x^1, y^2) and (x^2, y^1) are equilibrium points. Then $x_i^1 > 0$ implies $(A_i \cdot, y^2) = \max_k (A_k \cdot, y^2) = \alpha_2$, or $M_1(x^1) \subset M_2(y^2)$, and similarly, $x_i^2 > 0$ implies $(A_i \cdot, y^1) = \max_k (A_k \cdot, y^1) = \alpha_1$, or $M_1(x^2) \subset M_2(y^1)$. The remainder of the necessity aspects of the proof are clear.

Suppose, on the other hand, that the given conditions hold. The condition $M_1(x^1) \subset M_2(y^2)$ implies that if $(A_i \cdot, y^2) < \alpha_2$, then $x_i^1 = 0$, and the condition $N_1(y^2) \subset N_2(x^1)$ implies that if $(x^1, B \cdot_j) < \beta_1$ then $y_j^2 = 0$, from which it follows that (x^1, y^2) is an equilibrium point with payoffs α_2 and β_1 to the two players, respectively. The rest of the proof is identical in nature to that stated.

Partial interchangeability, e.g., (x^1, y^1) and (x^1, y^2) both equilibrium points but not (x^2, y^1) is also possible, and the conditions for this are clear. Generalization to the interchangeability of all equilibrium points (Nash-solvability) is also clear—one merely requires the conditions of Theorem 4 to hold for all pairs of equilibrium points.

IV. STABILITY OF EQUILIBRIUM POINTS

The stability of solutions to 0-sum matrix games with respect to change in the entries in the game matrix is a question which is easily handled by the sensitivity analysis techniques of linear programing. Since bimatrix games are not in general amenable to these techniques, it is of interest to be able to assess stability in this case also. For this purpose, by a "perturbation" A' of a real $m \times n$ matrix A will be meant another $m \times n$ real matrix.

THEOREM 5: Let $(\bar{x}, \bar{y})$ be an equilibrium point of a game (A, B) A, B $m \times n$, with payoffs α, β to P_1 and P_2, respectively. Then $(\bar{x}, \bar{y})$ will remain an equilibrium point of the game (A', B) obtained from the following perturbations of A:

(a) For all $i \in M_1(\bar{x})$, let $a_i \in E^n$ be such that $(a_i, \bar{y}) = c \neq 0$, otherwise arbitrary (real). Let $A_i' = A_i \cdot + a_i$. For $i \notin M_1(\bar{x})$, let $b_i \in E^n$ be such that $(A_i \cdot + b_i, \bar{y}) \leq \alpha + c$, and let $A_i' \cdot = A_i \cdot + b_i$.

(b) Let a_i^T be in the nullspace of $\bar{y}^T$, regarded as a linear transformation over E^n. Let $A_i' \cdot = A_i \cdot + a_i$ for any desired i. For $i \notin M_1(x)$, let $b_i \in E^n$ be such that $(A_i \cdot + b_i, \bar{y}) \leq \alpha$, and let $A_i' \cdot = A_i \cdot + b_i$.

Further, any perturbation of A leaving $(\bar{x}, \bar{y})$ as an equilibrium point of the perturbed game (A', B) must be one of the above.

PROOF: It is clear that any perturbation of the sort described will leave $(\bar{x}, \bar{y})$ as an equilibrium point of the perturbed game (A', B). Suppose A' is a matrix obtained from A by a perturbation, and suppose $(\bar{x}, \bar{y})$ is an equilibrium point of (A', B). Then, for $i \in M_1(\bar{x})$, $(A_i', \bar{y}) = \rho = \max_k (A_k' \cdot, \bar{y})$. If $\rho \neq \alpha$, let $a_i = A_i' \cdot - A_i \cdot$, let $c = \rho - \alpha$. For $i \notin M_1(\bar{x})$, let $b_i = A_i' - A_i \cdot$.

Since $(A_i' \cdot, \bar{y}) \leq \rho = \alpha + c$, $i \notin M_1(\bar{x})$, $(A_i \cdot + b_i, \bar{y}) \leq \alpha + c$ as in (a) above. If $\rho = \alpha$, for $i \in M_1(\bar{x})$ we have $(a_i, \bar{y}) = (A_i' \cdot - A_i \cdot, \bar{y}) = 0$, and a_i^T is in the nullspace of $\bar{y}^T$. If, for $i \notin M_1(\bar{x})$, $A_i' \cdot \neq A_i \cdot$, let $b_i = A_i' \cdot - A_i$. Since $(A_i' \cdot, \bar{y}) \leq \alpha$, $i \notin M_1(\bar{x})$, the perturbation A' of A is as in (b) above.

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AN OPTIMUM t-TEST FOR THE SCALE PARAMETER OF AN EXTREME-VALUE DISTRIBUTION

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ABSTRACT

A Student's t-test proposed by Ogawa is considered for the hypothesis $H_0: \sigma = \sigma_0$ against the alternative hypothesis $H_1: \sigma \neq \sigma_0$, where σ is the scale parameter of the Extreme-value distribution of smallest values with known location parameter μ. The test is based on a few sample quantiles chosen from a large sample so as to give asymptotically maximum power to the test when the number of sample quantiles is fixed. A table which facilitates the computation of the test statistic is given. Several schemes for determining the ranks of the sample quantiles by the optimal spacings are compared and the effect of the bias of the estimate of σ on the test is investigated through a Monte Carlo study.

1. INTRODUCTION

The set of $k > 2$ constants $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ such that $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < 1$ will be called a spacing. Define $n_i = [n\lambda_i] + 1 (i = 1, 2, \ldots, k)$, where $[n\lambda_i]$ is the greatest integer $\leqslant n\lambda_i$. Then the ordered observations

$$X_{(n_1)} < X_{(n_2)} < \ldots < X_{(n_k)}$$

taken from an ordered sample of size n,

$$X_{(1)} < X_{(2)} < \ldots < X_{(n)},$$

are called sample quantiles. In this paper the ordered sample is assumed to have been taken from the Extreme-value distribution of smallest values with distribution function

$$F\left(\frac{x-\mu}{\sigma}\right) = 1 - \exp\left[-\exp\left(\frac{x-\mu}{\sigma}\right)\right],$$

and density function

$$\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma} \exp\left[\left(\frac{x-\mu}{\sigma}\right) - \exp\left(\frac{x-\mu}{\sigma}\right)\right], -\infty < x < \infty, -\infty < \mu < \infty,$$

$$0 < \sigma < \infty,$$

where μ and σ are the location and scale parameters, respectively, and μ is assumed to be known. For convenience it is assumed that the known value for μ is $\mu=0$. Otherwise the known value of μ must be subtracted from the values of the sample quantiles.

It is well-known (cf. Sarhan and Greenberg [11] p. 17) that the joint distribution of the k sample quantiles is asymptotically normal as $n\to\infty$ with density function given by

$$h(x_{(n_1)}, \ldots, x_{(n_k)}) = g(\lambda_1, \ldots, \lambda_k, \sigma, n, k) \exp\left(-\frac{n}{2\sigma^2}\Omega\right),$$

where g is a function of $\lambda_1, \ldots, \lambda_k, \sigma, n, k$ and

$$\Omega=\Omega(\sigma)=\sum_{i=1}^{k}\frac{\lambda_{i+1}-\lambda_{i-1}}{(\lambda_{i+1}-\lambda_i)(\lambda_i-\lambda_{i-1})}f_i^2(x_{(n_i)}-\sigma u_i)^2-2\sum_{i=2}^{k}\frac{f_i f_{i-1}}{\lambda_i-\lambda_{i-1}}(x_{(n_i)}-\sigma u_i)(x_{(n_{i-1})}-\sigma u_{i-1}),$$

$f_i=f(u_i)$, $u_i=F^{-1}(\lambda_i)$ $(i=1, 2, \ldots, k)$, $\lambda_0=0$, $\lambda_{k+1}=1$. This paper presents a table which is useful for the calculation of a Student's t-test suggested by Ogawa [10] (see also Sarhan and Greenberg [11] p. 380) for testing the hypothesis H_0: $\sigma=\sigma_0$ against H_1: $\sigma\neq\sigma_0$. All calculations were performed on an IBM 7040 with output to eight decimal places, but rounded to four decimal places for the tables.

The usefulness of the Extreme-value distribution and the related Weibull distribution in life testing and reliability was brought to broad attention through the work of Weibull [12] and Lieblein and Zelen [5]. Subsequently, research on these distributions was extensively carried out. A brief history, detailed survey and interesting applications for results up to 1967 were given by Mann [7], Harter [2, 4] and Harter and Moore [3]. Mann [7] also systematically compared various point and interval estimation procedures.

Confidence bounds and tests of hypotheses concerning μ, σ, the reliability and the logarithm of the reliable life based on a few selected order statistics or the first (smallest) m order statistics have been studied by several authors for large and small sample sizes. Some interesting results were summarized by Mann [7]. More recently, Mann [9] (see also [6]) proposed a $(1-\alpha)$-level upper confidence bound based on two order statistics which gives the smallest risk among invariant two-order-statistic bounds when the sample is censored from above. Using two selected order statistics from each sample, Mann [8] also proposed upper and two-sided confidence bounds for the ratio of scale parameters from two Extreme-value distributions.

2. TESTING H_0: $\sigma=\sigma_0$

To test the null hypothesis,

$$H_0:\ \sigma=\sigma_0,$$

where σ_0 is a specified value of σ, against the alternative

$$H_1:\ \sigma\neq\sigma_0,$$

Ogawa [10] (see also Sarhan and Greenberg [11] p. 380) proposed the statistic

$$t=\frac{\sqrt{K_2}\left(\frac{\hat{\sigma}-\sigma}{\sigma}\right)+\delta}{\sqrt{\frac{\Omega_0}{(k-1)\sigma^2}}}.$$

where $\delta=\sqrt{K_2}\left(1-\frac{\sigma_0}{\sigma}\right)$, $K_2=\sum_{i=1}^{k+1}\frac{(f_iu_i-f_{i-1}u_{i-1})^2}{\lambda_i-\lambda_{i-1}}$, $f_0=f_{k+1}=0$ and $\hat{\sigma}$ is the value of σ which gives $\Omega(\sigma)$ its minimum value Ω_o ($\hat{\sigma}$ is called the asymptotically best linear estimate (ABLE) of σ when μ is known). If H_0 is true, then $\delta=0$ and the statistic t asymptotically has the Student t-distribution with $k-1$ degrees of freedom. If the alternative hypothesis is true, the statistic asymptotically has the noncentral t-distribution with noncentrality parameter $\delta\neq 0$. The power function of this t-test is an increasing function of the absolute value of δ and thus for a fixed number of sample quantiles the power of this test can be asymptotically maximized by choosing the spacing $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ which maximizes K_2. This spacing is called the optimum spacing. It was obtained by Chan and Kabir [1] for $k=1$ (1)10.

For computational purposes, it is useful to note that $\hat{\sigma}$ can be expressed as

$$\hat{\sigma}=\sum_{i=1}^{k} a_i x_{(n_i)},$$

where

$$a_i=\frac{f_i}{K_2}\left(\frac{f_iu_i-f_{i-1}u_{i-1}}{\lambda_i-\lambda_{i-1}}-\frac{f_{i+1}u_{i+1}-f_iu_i}{\lambda_{i+1}-\lambda_i}\right),$$

and the minimum Ω_0 of $\Omega(\sigma)$ can be expressed as

$$\Omega_0=\sum_{i=1}^{k+1}\frac{(f_ix_{(n_i)}-f_{i-1}x_{(n_{i-1})})^2}{\lambda_i-\lambda_{i-1}}-K_2\hat{\sigma}^2$$

$$=\omega_o-K_2\hat{\sigma}^2,$$

where

$$\omega_0=\sum_{i=1}^{k}(b_ix_{(n_i)}^2+c_ix_{(n_i)}x_{(n_{i+1})}),$$

$$b_i=f_i^2\left(\frac{1}{\lambda_i-\lambda_{i-1}}+\frac{1}{\lambda_{i+1}-\lambda_i}\right),$$

$$c_i=-\frac{2f_if_{i+1}}{\lambda_{i+1}-\lambda_i}.$$

Table 1 gives the values of the optimal λ_i and the corresponding a_i, b_i, c_i ($i=1, 2, \ldots, k$) and K_2 for $k=2$(1)10.

3. AN EXAMPLE

Consider the following ordered random sample of size 30 generated from an Extreme-value distribution with location parameter $\mu=2$ and scale parameter $\sigma=1$:

−0.6366, −0.3658, 0.0095, 0.1343, 0.1522, 0.1838, 0.2299, 0.5751, 1.1546, 1.2231, 1.3276, 1.3594, 1.5329, 1.5858, 1.7465, 1.7496, 1.9434, 2.0312, 2.1740, 2.1873, 2.2552, 2.2670, 2.3577, 2.8664, 2.9255, 2.9829, 3.0326, 3.0979, 3.2884, 3.4241.

TABLE 1. *The optimum spacings* λ_i *for the Extreme-value distribution of smallest values, the coefficients* a_i *for the calculation of* $\hat{\sigma}$, *the coefficients* b_i, c_i *for the calculation of* ω_0, *and the values of* K_2.

k	λ_1 a_1 b_1 c_1	λ_2 a_2 b_2 c_2	λ_3 a_3 b_3 c_3	λ_4 a_4 b_4 c_4	λ_5 a_5 b_5 c_5	λ_6 a_6 b_6 c_6	λ_7 a_7 b_7 c_7	λ_8 a_8 b_8 c_8	λ_9 a_9 b_9 c_9	λ_{10} a_{10} b_{10} c_{10}	K_2
2	0.1322 −0.2364 0.1328 −0.0364	0.9627 0.4521 0.4216 0.0000									1.1886
3	0.1397 −0.2175 0.1411 −0.0595	0.9324 0.3246 0.6217 −0.3024	0.9896 0.1754 0.2562 0.0000								1.3643
4	0.0503 −0.0703 0.0619 −0.1119	0.2201 −0.1776 0.2745 −0.1048	0.9270 0.3050 0.6414 −0.3084	0.9889 0.1668 0.2652 0.0000							1.5018
5	0.0520 −0.0694 0.0639 −0.1151	0.2274 −0.1743 0.2853 −0.1334	0.9031 0.2337 0.8107 −0.6389	0.9727 0.1734 0.5627 −0.2097	0.9955 0.0832 0.1573 0.0000						1.5755
6	0.0242 −0.0307 0.0304 −0.0580	0.1090 −0.0837 0.1873 −0.2862	0.2780 −0.1416 0.4163 −0.1755	0.8989 0.2292 0.8238 −0.6421	0.9717 0.1711 0.5711 −0.2154	0.9953 0.0828 0.1619 0.0000					1.6294
7	0.0248 −0.0307 0.0311 −0.0594	0.1114 −0.0835 0.1910 −0.2908	0.2841 −0.1408 0.4276 −0.2043	0.8800 0.1840 0.9706 −0.9441	0.9551 0.1566 0.8788 −0.5204	0.9864 0.1049 0.4115 −0.1445	0.9977 0.0473 0.1022 0.0000				1.6671
8	0.0134 −0.0162 0.0169 −0.0329	0.0619 −0.0455 0.1111 −0.1799	0.1589 −0.0828 0.3506 −0.4755	0.3191 −0.1173 0.5503 −0.2423	0.8767 0.1824 0.9800 −0.9432	0.9541 0.1560 0.8835 −0.5254	0.9861 0.1043 0.4176 −0.1496	0.9976 0.0478 0.1056 0.0000			1.6938
9	0.0136 −0.0162 0.0171 −0.0334	0.0628 −0.0456 0.1126 −0.1820	0.1614 −0.0830 0.3549 −0.4804	0.3241 −0.1170 0.5613 −0.2698	0.8615 0.1514 1.1118 −1.2168	0.9386 0.1395 1.1762 −0.8438	0.9755 0.1072 0.7153 −0.4054	0.9923 0.0678 0.3065 −0.1094	0.9986 0.0306 0.0739 0.0000		1.7156
10	0.0082 −0.0096 0.0103 −0.0204	0.0384 −0.0273 0.0700 −0.1161	0.0998 −0.0514 0.2355 −0.3376	0.1998 −0.0781 0.5301 −0.6653	0.3499 −0.0996 0.6761 −0.3041	0.8588 0.1506 1.1197 −1.2144	0.9376 0.1392 1.1794 −0.8490	0.9751 0.1074 0.7199 −0.4072	0.9922 0.0680 0.3078 −0.1088	0.9986 0.0305 0.0737 0.0000	1.7308

Now assume that $\mu=2$ is known and the hypothesis,

$$H_0\colon\ \sigma=1,$$

is to be tested by using $k=3$ sample quantiles. By using the optimum spacing {0.1397, 0.9324, 0.9896}

from Table 1, it can be calculated that $n_1=5$, $n_2=28$, $n_3=30$. Thus, subtracting $\mu=2$ from these three observations, we find that the three sample quantiles are

$$x_{(5)}=-1.8478,\ x_{(28)}=1.0979,\ x_{(30)}=1.4241.$$

From Table 1,

$$\hat{\sigma}=(-0.2175)(-1.8478)+(0.3246)(1.0979)+(0.1754)(1.4241)=1.0081,$$

$$\omega_0=(0.1411)(-1.8478)^2+(0.6217)(1.0979)^2+(0.2562)(1.4241)^2$$
$$+(-0.0595)(-1.8478)(1.0979)+(-0.3024)(1.0979)(1.4241)=1.3986$$

and

$$\Omega_0=1.3986-(1.3643)(1.0081)^2$$
$$=0.0121.$$

Therefore

$$t=\sqrt{1.3643}\,\frac{(1.0081-1.0000)}{\sqrt{\frac{0.0121}{2}}}$$
$$=0.1216.$$

From the table of the t-distribution with 2 degrees of freedom, the critical value for a two tailed test at the 5-percent level of significance is 4.303. Thus the hypothesis,

$$H_0\colon \sigma=1,$$

is accepted.

The computation of t should be performed carefully because the numerator and denominator often assume small values, and a minor error in either might result in a large error in the value of t.

4. THE EFFECTS ON THE TEST OF DIFFERENT SCHEMES FOR DETERMINING RANKS AND OF THE BIASES OF ESTIMATES OF σ

The effects of the schemes and biases in the test were investigated through a Monte Carlo study. 10,000 samples of sizes $n=20$ and 100 were generated.

The scheme for determining the ranks of the selected order statistics used in the preceding sections is the traditional scheme, i.e.,

SCHEME I: $[n\lambda_i]+1$.

This scheme will be compared with three others:

SCHEME II: $[n\lambda_i]$.

SCHEME III: $x_{(n\lambda_i)}$ is defined to be $X_{[n\lambda_i]}+(n\lambda_i-[n\lambda_i])(X_{[n\lambda_i]+1}-X_{[n\lambda_i]})$; i.e., the result of linear interpolation between the adjacent order statistics $X_{[n\lambda_i]}$ and $X_{[n\lambda_i]+1}$ (extrapolation is required if $n\lambda_i<1$, in which case $X_{(n\lambda_i)}=X_{[n\lambda_i]+1}-\{([n\lambda_i]+1)-n\lambda_i\}(X_{[n\lambda_i]+2}-X_{[n\lambda_i]+1})$).

TABLE 2. *The significance levels* $(\alpha_L^* + \alpha_R^*)$ *and powers* $1-\beta_\alpha^*$ *obtained from simulating the exact distributions of* t_1, t_2, t_3, *and* t_4 *determined by Schemes I, II, III, and IV and their debiased versions when their limit distribution has significance level* $(\alpha/2+\alpha/2)$ *and power* $1-\beta_\alpha$.

			Original Version								Debiased Version								
			$\alpha=0.01$		$1-\beta_\alpha$		$\alpha=0.05$		$1-\beta_\alpha$		$\alpha=0.01$		$1-\beta_\alpha$		$\alpha=0.05$		$1-\beta_\alpha$		$\frac{K_2}{K_2^{(i)}}$
					0.01	0.50			0.01	0.50			0.01	0.50			0.01	0.50	
n	k	Scheme	α_L^*	α_R^*	$1-\beta_\alpha^*$		α_L^*	α_R^*	$1-\beta_\alpha^*$		α_L^*	α_R^*	$1-\beta_\alpha^*$		α_L^*	α_R^*	$1-\beta_\alpha^*$		
20	3	I	0.008	0.001	0.989	0.561	0.048	0.008	0.991	0.518	0.010	0.006	0.995	0.677	0.049	0.029	0.996	0.663	1.053
	3	IV	0.003	0.008	0.994	0.614	0.020	0.038	0.996	0.630	0.011	0.004	0.994	0.651	0.049	0.026	0.995	0.638	0.944
	4	I	0.020	0.001	0.996	0.548	0.089	0.003	0.996	0.448	0.017	0.005	0.997	0.723	0.064	0.034	0.998	0.690	1.104
	4	IV	0.003	0.008	0.976	0.600	0.022	0.039	0.985	0.618	0.011	0.003	0.981	0.546	0.047	0.018	0.987	0.525	0.928
100	3	I	0.005	0.003	0.993	0.452	0.028	0.014	0.994	0.449	0.006	0.005	0.993	0.567	0.032	0.022	0.995	0.560	1.008
	3	II	0.006	0.001	0.975	0.309	0.026	0.004	0.977	0.284	0.006	0.004	0.995	0.606	0.034	0.029	0.997	0.595	1.029
	3	III	0.007	0.003	0.992	0.486	0.038	0.013	0.993	0.461	0.006	0.004	0.995	0.573	0.031	0.025	0.996	0.564	1.021
	3	IV	0.003	0.007	0.983	0.508	0.017	0.030	0.990	0.540	0.005	0.004	0.987	0.477	0.024	0.019	0.991	0.474	0.973
	4	I	0.012	0.002	0.993	0.473	0.055	0.007	0.993	0.426	0.008	0.004	0.995	0.565	0.035	0.022	0.995	0.557	1.032
	4	II	0.011	0.000	0.972	0.349	0.054	0.003	0.976	0.304	0.008	0.005	0.993	0.569	0.036	0.025	0.995	0.558	1.045
	4	III	0.008	0.002	0.985	0.477	0.038	0.012	0.989	0.453	0.008	0.004	0.993	0.550	0.033	0.022	0.994	0.538	1.015
	4	IV	0.004	0.007	0.988	0.538	0.017	0.035	0.994	0.572	0.007	0.004	0.989	0.480	0.026	0.018	0.992	0.475	0.971
	5	I	0.009	0.002	0.996	0.527	0.038	0.018	0.996	0.514	0.008	0.003	0.996	0.564	0.034	0.025	0.997	0.551	1.005
	5	II	0.009	0.001	0.982	0.396	0.045	0.005	0.986	0.370	0.009	0.005	0.988	0.539	0.033	0.024	0.991	0.528	1.025
	5	III	0.013	0.001	0.992	0.505	0.054	0.010	0.993	0.467	0.012	0.004	0.997	0.624	0.046	0.027	0.997	0.596	1.018
	5	IV	0.004	0.006	0.990	0.558	0.018	0.033	0.994	0.576	0.007	0.003	0.988	0.492	0.030	0.018	0.992	0.487	0.978

SCHEME IV: $X_{(n+1)\lambda_i}$ is defined to be $X_{[(n+1)\lambda_i]}+\{(n+1)\lambda_i-[(n+1)\lambda_i]\}\,(X_{[(n+1)\lambda]i+1}-X_{[(n+1)\lambda_i]})$ (extrapolation is required if $(n+1)\lambda_i<1$ or $(n+1)\lambda_i>n$). The values of $(\hat{\sigma}, t)$ corresponding to Schemes II, III, and IV will be denoted by $(\hat{\sigma}_2, t_2)$, $(\hat{\sigma}_3, t_3)$ and $(\hat{\sigma}_4, t_4)$, respectively.

Let α be the significance level, $t(k-1;\ \alpha)>0$ the two-tailed critical value and $1-\beta_\alpha$ the power of the test when t (or t_2, t_3, t_4) is assumed to have its limiting central or noncentral Student's t-distribution with $k-1$ degrees of freedom. Some simulated values of the probabilities

$$\alpha_L^*=P(t<-t(k-1;\ \alpha)),\ \alpha_R^*=P(t>t(k-1;\ \alpha)) \text{ corresponding to } \alpha_L=\alpha_R=\alpha/2,$$

and

$$1-\beta_\alpha^*=P(|t|>t(k-1;\ \alpha)\,||\delta|) \text{ corresponding to } 1-\beta_\alpha$$

for the exact distribution of t and similar values for t_2, t_3, and t_4 are listed in Table 2 under the subheading "Original Version".

For finite n, $\hat{\sigma}$ is not unbiased. The expectation $\mathscr{E}\hat{\sigma}\neq\sigma$ and hence it may be suspected that the bias will have some effect on the test statistic, t. The bias affects both the numerator and denominator of t. It can easily be proved that the estimate

$$\hat{\sigma}'=(K_2/K_2')\hat{\sigma}$$

is strictly unbiased for any n and k, where

$$K_2'=\sum_{i=1}^{k+1}\frac{(f_iu_i-f_{i-1}u_{i-1})\,(f_ie_{n_i}-f_{i-1}e_{n_i})}{\lambda_i-\lambda_{i-1}},\quad e_{n_i}=\mathscr{E}\left(\frac{X_{(n_i)}-\mu}{\sigma}\right).$$

If $\hat{\sigma}$ in the numerator of t is changed to $\hat{\sigma}'$, and if Ω_0 in the denominator is changed to

$$\Omega_0'=\sum_{i=1}^{k}\frac{\lambda_{i+1}-\lambda_{i-1}}{(\lambda_{i+1}-\lambda_i)(\lambda_i-\lambda_{i-1})}f_i^2(x_{(n_i)}-\hat{\sigma}'e_{n_i})^2-2\sum_{i=2}^{k}\frac{f_if_{i-1}}{\lambda_i-\lambda_{i-1}}(x_{(n_i)}-\hat{\sigma}'e_{n_i})(x_{(n_{i-1})}-\hat{\sigma}'e_{n_{i-1}})$$

(hence $\mathscr{E}(X_{(n_i)}-\hat{\sigma}'e_{n_i})=0$), then the test statistic t will be called a debiased version of t.

Let K_2'', K_2''', and K_2'''' be the same as K_2' except that the expectations e_{n_i} are changed to the expectations of the order statistics determined by Schemes II, III, and IV. So for example, when Scheme IV is used,

$$\mathscr{E}\left(\frac{X_{((n+1)\lambda_i)}-\mu}{\sigma}\right)=e_{[(n+1)\lambda_i]}+\{(n+1)\lambda_i-[(n+1)\lambda_i]\}(e_{[(n+1)\lambda_i]+1}-e_{[(n+1)\lambda_i]}).$$

Then

$$\hat{\sigma}''=(K_2/K_2'')\hat{\sigma}_2,\ \hat{\sigma}'''=(K_2/K_2''')\hat{\sigma}_3 \text{ and } \hat{\sigma}''''=(K_2/K_2'''')\hat{\sigma}_4$$

are also unbiased for every n. Debiased versions of t_2, t_3, and t_4 similar to that of t can then be obtained.

The values of K_2/K_2', K_2/K_2'', K_2/K_2''', and K_2/K_2'''' are given in Table 2.

Some simulated values of α_L^*, α_R^*, and $1-\beta_\alpha^*$ of the debiased versions of t, t_2, t_3, and t_4 are given in Table 2 for comparison with the original version. For each sample size n, 10,000 random samples were generated in the I.B.M. 7040 computer.

From Table 2 and some similar simulated results not listed there, the following conclusions may be drawn.

(1) When the expectations $e_{n_i}=\mathscr{E}((X_{(n_i)}-\mu)/\sigma)$ are available (for $n \leq 100$ they were computed by White [13]), the debiased version should be used. Scheme I has the advantage of its computational simplicity (no interpolation or extrapolation of $X_{(n_i)}$ and e_{n_i} is required). But compared with other schemes its values of α_L^*, α_R^*, and $1-\beta_\alpha^*$ are not always closer to the corresponding α_L, α_R, and $1-\beta_\alpha$.

(2) When the expectations e_{n_i} are not available, Scheme IV should be used if n is of intermediate size. If n is very large, the difference between Schemes I and IV will be small and hence Scheme I is preferable to Scheme IV due to its computational simplicity.

(3) The values of K_2/K_2'' tend to be somewhat closer to 1 then those of K_2/K_2', K_2 K_2'' and K_2/K_2''''.

It seems reasonable to predict that, in general, Scheme IV provides a more logical determination of the positions of sample quantiles in a sample. A thorough summary and discussion of various schemes for determining sample quantiles in estimation has recently been published by Harter [4].

If in addition to changing $\hat{\sigma}_4$ to $\hat{\sigma}''''$ in t_4, all the u_i appearing in $f_i=f(u_i)$ and $\lambda_i=F(u_i)$ in Ω_o and K_2 are changed to the corresponding $\mathscr{E}((X_{((n+1)\lambda_i)}-\mu)/\sigma)$, the procedure may be better since it eliminates all bias. However, the computation of t_4 becomes much more tedious.

5. ACKNOWLEDGEMENT

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SCHEDULING DATA TRANSMISSION UNDER AN $\{S_i; 0\}$ POLICY

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ABSTRACT

This paper discusses scheduling of data transmission when data can only be transmitted in one direction at a time. A common policy used is the so-called alternating priority policy. In this paper we select a more general class of policies named the $\{S_i; 0\}$ policy. We show how to determine the optimal parameters of the $\{S_i; 0\}$ policy for given system parameters. We also give a simple example to show that $\{S_i; 0\}$ policy is, in fact, better then alternating priority policy.

1. INTRODUCTION

In data transmission one is faced many times by a situation in which data can only be transmitted in one direction at a time. This type of transmission is called half duplex. The data keeps accumulating on side X while transmission from side Y to X is underway. Everytime a switch is made in the direction of transmission, from one side to the other, a finite switching time is incurred. In this paper we discuss an $\{S_i; 0\}$ policy for optimal transmission of data.

A common policy that is used in practice is the so-called alternating priority policy [1, 2]. Under this policy one keeps transmitting data from X to Y until the backlog of messages at X reaches zero. One then switches to Y and transmits data from Y to X. Under an $\{S_i; 0\}$ policy, data are transmitted from X to Y until the backlog of data at X reaches zero and the backlog of data at Y is at least S_Y. If the backlog at Y has not reached S_Y the direction of transmission is still kept as X to Y until the above mentioned two conditions are satisfied.

In this paper, we show how to calculate the optimal values of the policy parameters given the arrival and service rates of the data to be transmitted. Since alternating priority policy is a special case of $\{S_i; 0\}$ policy it is dominated by the $\{S_i; 0\}$ policy. We also calculate the values of mean delay under alternating priority policy and the $\{S_i; 0\}$ policy for a particular example and show that $\{S_i; 0\}$ policy is, in fact, better.

Although the results are presented using the terminology for the above example they are equally applicable to scheduling of labor limited job shops and problems involving sharing of server [3].

2. NOTATIONS AND ASSUMPTIONS

A. Single server with two queues accumulating messages to be transmitted in opposite directions.

B. At any time, the messages can be transmitted in one direction only.

C. The objective is to minimize the mean delay.

D. The arrival process of messages is assumed to be continuous arrivals with mean arrival rate at queue $i = \lambda_i$, $i = 1, 2$. By continuous arrivals we mean that the number of message units that have ar-

rived in a given period of time increases linearly with time instead of at discrete points as shown in Figure 1.

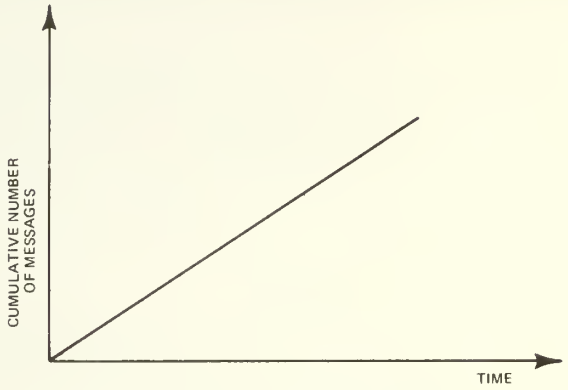

FIGURE 1. Continuous arrivals

E. The service process of messages is assumed to be continuous service with mean service rates at queue $i = \mu_i$, $(i = 1, 2)$ when there is a backlog of messages to be transmitted. By continuous service we mean that the number of message units waiting to be transmitted decreases continuously with time units instead of at discrete points as shown in Figure 2.

F. Time taken to switch service from direction i to direction j is given by $\zeta_{ij}(i, j = 1, 2, i \neq j)$.

G. $\rho_i = \dfrac{\lambda_i}{\mu_i}$ = utilization ratio, $i = 1, 2$.

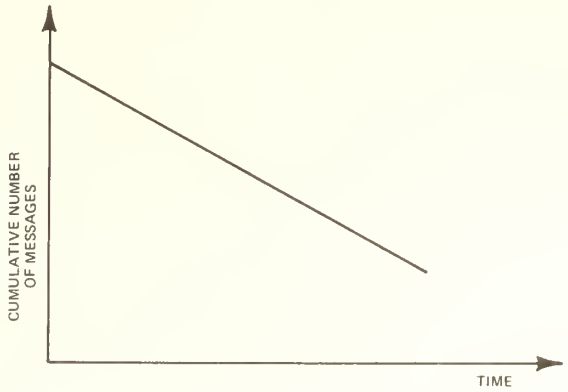

FIGURE 2. Continuous service

H. $\rho_1 + \rho_2 < 1$, i.e., the server is capable of handling the transmitting load. (Please see appendix.)

3. $\{S_i; 0\}$ POLICY-MODEL FORMULATION

In this section we determine the optimum length of the cycle time, T, and the optimal value of S_1 and S_2 assuming a stationary $\{S_i; 0\}$ policy. We will refer to Figure 3 throughout this section. In Figure 3 the server reduces the backlog of messages at queue 1 to zero at point C, but continues to work at queue 1 until the point E at which point the backlog of messages at queue 2 has built to S_2 units. At point E the server switches to queue 2. The time taken to switch (because of the overhead of switching) from queue 1 to queue 2 is shown as EM in Figure 3 and is equal to ζ_{12}. During this period the queue 2 continues to

build up because the server is involved in a nonproductive function of switching. At point M the server begins transmitting messages from queue 2.

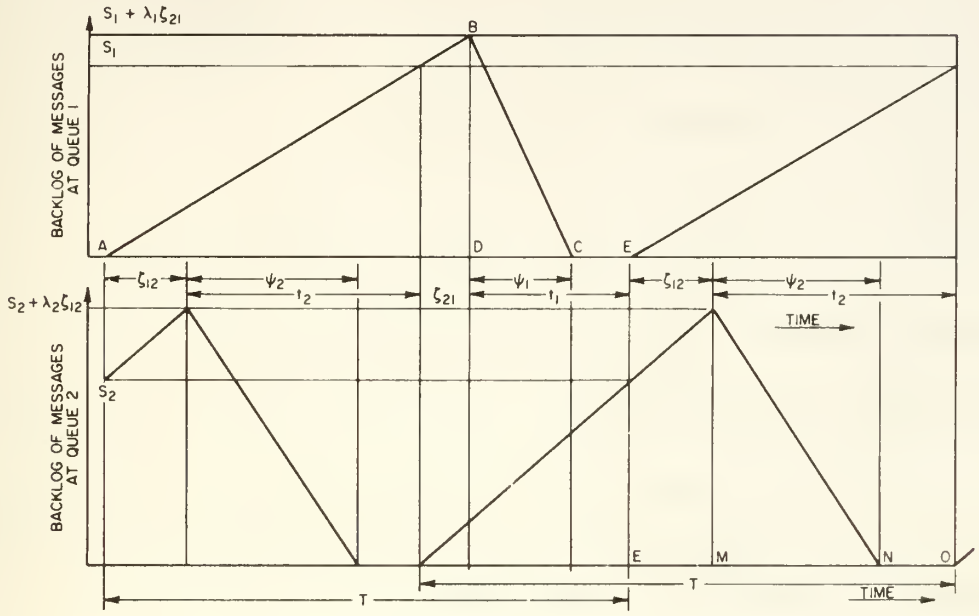

FIGURE 3. $\{S_i; 0\}$ Policy

Let

W_1 = Total delay time of messages in a cycle of time T at queue 1

= Area under the curve ABC

$$= (1/2)(S_1+\lambda_1\zeta_{21})(T-t_1)+(1/2)(S_1+\lambda_1\zeta_{21})\Psi_1.$$

Now

$S_1+\lambda_1\zeta_{21}$ = Total number of messages that arrive during time $(T-t_1)$ at queue 1 $=\lambda_1(T-t_1)$

$\mu_1\Psi_1$ = Total number of messages serviced at queue 1 during time Ψ_1

= Total number of messages that arrived during time $(T-t_1+\Psi_1)$

$$= \lambda_1(T-t_1+\Psi_1)$$

or

$$\mu_1\Psi_1 = \lambda_1 T-\lambda_1 t_1+\lambda_1\Psi_1$$

or

$$\Psi_1 = \frac{\lambda_1(T-t_1)}{\mu_1-\lambda_1}. \tag{2}$$

Substituting the values of $(S_1+\lambda_1\zeta_{21})$ and Ψ_1 in Equation (1) we have

$$W_1=(1/2)\lambda_1(T-t_1)(T-t_1)+(1/2)\lambda_1(T-t_1)\frac{\lambda_1(T-t_1)}{\mu_1-\lambda_1}$$

$$=(1/2)\lambda_1(T-t_1)^2+(1/2)\lambda_1^2\frac{(T-t_1)^2}{\mu_1-\lambda_1}$$

$$=(1/2)\lambda_1(T-t_1)^2\left(1+\frac{\lambda_1}{\mu_1-\lambda_1}\right)=(1/2)\frac{\lambda_1\mu_1}{\mu_1-\lambda_1}(T-t_1)^2.$$

Total number of messages that arrived at Queue 1 during time $T=\lambda_1 T$.

Mean waiting time of messages at Queue 1

$$WM_1=\frac{W_1}{\lambda_1 T}=(1/2)\frac{\mu_1}{(\mu_1-\lambda_1)T}(T-t_1)^2.$$

Similarly the mean waiting time of messages at Queue 2

$$=WM_2=(1/2)\frac{\mu_2}{(\mu_2-\lambda_2)T}(T-t_2)^2.$$

In general mean delay time/message

$$W_M=\frac{\lambda_1 T(WM_1)+\lambda_2 T(WM_2)}{(\lambda_1+\lambda_2)T}$$

$$=\frac{\lambda_1(WM_1)+\lambda_2(WM_2)}{(\lambda_1+\lambda_2)}$$

$$=(1/2)\frac{\lambda_1\mu_1(T-t_1)^2}{(\lambda_1+\lambda_2)(\mu_1-\lambda_1)T}+(1/2)\frac{\lambda_2\mu_2(T-t_2)^2}{(\lambda_1+\lambda_2)(\mu_2-\lambda_2)T}$$

$$=(1/2)\frac{\lambda_1\mu_1 T}{(\lambda_1+\lambda_2)(\mu_1-\lambda_1)}\left(1-\frac{t_1}{T}\right)^2+(1/2)\frac{\lambda_2\mu_2 T}{(\lambda_1+\lambda_2)(\mu_2-\lambda_2)}\left(1-\frac{t_2}{T}\right)^2.$$

Let

$$c_1=(1/2)\frac{\mu_1\lambda_1}{(\lambda_1+\lambda_2)(\mu_1-\lambda_1)}$$

and

$$c_2=(1/2)\frac{\mu_2\lambda_2}{(\lambda_1+\lambda_2)(\mu_2-\lambda_2)}.$$

Then $W_M=c_1T\left(1-\frac{t_1}{T}\right)^2+c_2T\left(1-\frac{t_2}{T}\right)^2.$

We now want to minimize the mean delay time of messages (W_M), but there are certain conditions to be satisfied as described below:

From Figure (3) we have

$$\Psi_1 \leq t_1$$

or by use of Equation (2) we have

$$\frac{\lambda_1(T-t_1)}{\mu_1-\lambda_1} \leq t_1$$

$$\frac{T-t_1}{t_1} \leq \frac{\mu_1-\lambda_1}{\lambda_1} \quad \text{or} \quad \frac{t_1}{T} \geq \frac{\lambda_1}{\mu_1}.$$

Similarly

$$\Psi_2 \leq t_2 \Longrightarrow \frac{\lambda_2}{\mu_2} \leq \frac{t_2}{T}$$

Also we have

$$t_1+t_2+\zeta_{12}+\zeta_{21}=T.$$

Let $\zeta_{12}+\zeta_{21}=\zeta$.

Then our problem is to find the value of t_1, t_2, T so as to

$$\text{minimize } W_M=c_1T\left(1-\frac{t_1}{T}\right)^2+c_2T\left(1-\frac{t_2}{T}\right)^2$$

subject to:

$$\frac{\lambda_1}{\mu_1}=\rho_1 \leq \frac{t_1}{T}$$

$$\frac{\lambda_2}{\mu_2}=\rho_2 \leq \frac{t_2}{T}$$

$$\frac{t_1}{T}+\frac{t_2}{T}+\frac{\zeta}{T}=1$$

$$t_1,\ t_2,\ T \geq 0,$$

or

$$\min W_M=c_1T\left(1-\frac{t_1}{T}\right)^2+c_2T\left(1-\frac{t_2}{T}\right)^2$$

subject to:

(4)

$$\rho_1T-t_1 \leq 0$$

$$\rho_2T-t_2 \leq 0$$

$$t_1+t_2+\zeta=T$$

$$t_1,\ t_2,\ T \geq 0.$$

4. PROBLEM SOLUTION

We now prove a theorem, which simplifies Problem (4) and enables us to obtain the optimal solution explicitly, in a closed form.

THEOREM 1: To minimize the mean delay time of messages under the following assumptions: (Reference Figure 3).

A. Assumptions made in section 2.

B. $\{S_i; 0\}$ Policy.

C. $\lambda_1 > \lambda_2$

D. T, the cycle time is fixed and $T \geq T_{min} = \Psi_1 + \Psi_2 + \zeta_{12} + \zeta_{21}$.

The server should apportion the cycle time, T, so that

$$t_2 = \Psi_2 \text{ and } t_1 = T - (t_2 + \zeta_{12} + \zeta_{21}) > \Psi_1.$$

PROOF: When the server is working at queue 2, and all the messages in the queue have been transmitted, say at point N, the server has a choice of switching immediately to queue 1 or of staying at queue 2. In the second alternative, the server waits at queue 2 for more messages to arrive and transmits them as they arrive, and finally switches to queue 1 at point 0 (in Figure 3). This theorem says that if $\lambda_1 > \lambda_2$, it is always advantageous to switch to queue 1 immediately at point N or that $\Psi_2 = t_2$.

If $T = T_{min}$ then it is obvious that $t_2 = \Psi_2$ and $t_1 = \Psi_1$ because the server has to finish the messages at the present queue before leaving for the next queue.

If $T > T_{min}$, the server can spend $(T - T_{min})$ units of time at queue 1 or queue 2 or both. In either of the above cases this time will be spent at a queue by the server only after he has reduced the backlog of messages to zero at that particular queue.

If the server spends this extra time at queue 1 the rate of increase of number of messages in the system is λ_2. On the other hand, if the server spends this extra time at queue 2 the rate of increase of number of messages in the system is λ_1.

Since $\lambda_1 > \lambda_2$ by assumption, the server should spend this extra time at queue 1. This implies that $\Psi_2 = t_2$ and $t_1 = T - (\zeta_{21} + \Psi_2 + \zeta_{12})$. Q.E.D.

Now from Equation (3) we have

$$\Psi_2 = t_2 \Longrightarrow \frac{\lambda_2}{\mu_2} = \frac{t_2}{T} \Longrightarrow \frac{t_2}{T} = \rho_2.$$

Problem (4) then reduces to

$$\text{Min } W_M = c_1 T \left(1 - \frac{t_1}{T}\right)^2 + c_2 T (1 - \rho_2)^2$$

subject to:

$$\rho_1 \leq \frac{t_1}{T} \qquad \text{and } \frac{t_1}{T} + \rho_2 + \frac{\zeta}{T} = 1 \qquad \text{or } 1 - \frac{t_1}{T} = \frac{\zeta}{T} + \rho_2$$

or Problem (4) can be rewritten as

$$\text{Min } W_M = c_1 T \left(\frac{\zeta}{T} + \rho_2\right)^2 + c_2 T(1-\rho_2)^2$$

subject to:

$$\rho_1 \leq 1 - \frac{\zeta}{T} - \rho_2 \qquad \text{or} \quad T \geq \frac{\zeta}{1-\rho_1-\rho_2},$$

or we have

$$\text{Min } W_M = c_1 T \left(\frac{\zeta}{T} + \rho_2\right)^2 + c_2 T(1-\rho_2)^2$$

subject to:

(5)
$$T \geq \frac{\zeta}{1-\rho_1-\rho_2}.$$

Problem (5) is a single variable and a single constrained optimization problem. To solve Problem (5) we first find the unconstrained minimum of W_M. We can easily show that W_M is a convex function. Now,

$$\frac{\partial W_M}{\partial T} = c_1 \left[-\frac{\zeta^2}{T^2} + \rho_2^2\right] + c_2[1-\rho_2]^2.$$

To determine the optimal value of cycle length, T, denote by T^* we equate $\dfrac{\partial W_M}{\partial T} = 0$ at $T = T^*$

or
$$c_1 \left[-\frac{\zeta^2}{(T^*)^2} + \rho_2^2\right] + c_2[1-\rho_2^2] = 0$$

or
$$(T^*)^2 = \frac{c_1\zeta^2}{c_1\rho_2^2 + c_2[1-\rho_2^2]}.$$

Then T^*, the optimal cycle length, can be found using the following rule:

$$\left\langle \frac{c_1\zeta^2}{c_1\rho_2^2 + c_2(1-\rho_2)^2} \leq \frac{\zeta^2}{(1-\rho_1-\rho_2)^2} \right\rangle \implies \left\langle T^* = \frac{\zeta}{1-\rho_1-\rho_2} \right\rangle$$

(6)
$$\left\langle \frac{c_1\zeta^2}{c_1\rho_2^2 + c_2(1-\rho_2)^2} > \frac{\zeta^2}{(1-\rho_1-\rho_2)^2} \right\rangle \implies \left\langle T^* = \left[\frac{c_1\zeta^2}{c_1\rho_2^2 + c_2(1-\rho_2)}\right]^{1/2} \right\rangle.$$

Once we have found T^*, t_1^*, and t_2^* can be obtained as follows:

$$t_2^* = \rho_2 T^*$$

$$t_1^* = T^* - \zeta - t_2^*$$

$$S_1^* = \lambda_1(T^* - t_1^*) - \lambda_1\zeta_{21} = \lambda_1(T^* - t_1^* - \zeta_{21})$$

$$S_2^* = \lambda_2(T^* - t_2^*) - \lambda_2\zeta_{12} = \lambda_2(T^* - t_2^* - \zeta_{12}).$$

Note, from Equation (6) we have that, whether the server should switch immediately to the other direction on reducing the backlog of messages to zero at the present queue or stay at the present queue is independent of the magnitude of ζ given that $\zeta > 0$. The decision is dependent upon ρ_1, ρ_2 and the ratio of λ_2/λ_1.

We have shown above that the optimum value of cycle time T can be calculated by one of the two expressions for T given the arrival and service rates at machines 1 and 2. Now we present Table 1, which gives the critical values of λ_1/λ_2 for different values of ρ_1 and ρ_2. If the actual values of λ_2/λ_1 in a given situation is less than or equal to this critical value, then the optimal value of the cycle time is given by the expression (7) otherwise the optimum cycle time is given by expression (8).

(7) $$T^* = \left[\frac{c_1\zeta^2}{c_1\rho_2^2 + c_2[1-\rho_2]^2}\right]^{1/2}$$

(8) $$T^* = \frac{\zeta}{1-\rho_1-\rho_2}.$$

TABLE 1. *Critical Values of λ_2/λ_1 for Different Values of ρ_2 and ρ_1*

ρ_1	ρ_2									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	0.778	0.625	0.429	0.167	0.000	0.000	0.000	0.000	0.000	0.000
0.2	0.667	0.500	0.286	0.000	↓	↓	↓	↓	↓	↓
0.3	0.556	0.375	0.143	0.000						
0.4	0.444	0.250	0.000	↓						
0.5	0.333	0.125	0.000							
0.6	0.222	0.000	↓							
0.7	0.111	0.000								
0.8	0.000	↓								
0.9	↓									
1.0										

If the calculated value of λ_2/λ_1 is less than the critical value of λ_2/λ_1

$$T^* = \left[\frac{c_1\zeta^2}{c_1\rho_2^2 + c_2\,[1-\rho_2]^2}\right]^{1/2}, \text{ otherwise } T^* = \frac{\zeta}{1-\rho_1-\rho_2}.$$

5. DISCUSSION

In the preceding sections we have derived the optimal value of parameters of the $\{S_i; 0\}$ policy in terms of the arrival and service rates of the messages. In the case of alternating priority $S_i = 0$ for $i = 1, 2$. We now show by taking a simple example that $\{S_i; 0\}$ policy is, in fact, superior to the alternating priority policy.

Let

$$\lambda_1 = 4,\ \lambda_2 = 1$$

$$\mu_1 = 20,\ \mu_2 = 5,$$

then

$$\rho_1 = 1/5, \rho_2 = 1/5.$$

A. Alternating Priority Policy

$$T^* = \frac{1}{1 - \rho_1 - \rho_2} = \frac{\zeta}{1 - 1/5 - 1/5} = 5/3\ \zeta.$$

$$\text{Mean Delay} = \sum_{i=1}^{2} (1/2)\ \frac{\lambda_i}{\lambda_1 + \lambda_2}\ (1 - \rho_i) T^*$$

$$= 0.67\ \zeta.$$

B. $\{S_i; 0\}$ Policy

$$T^* = \left[\frac{c_1 \zeta^2}{c_1 \rho_2^2 + c_2 (1 - \rho_2)^2}\right]^{1/2} = 2.05\zeta$$

$$W_M = c_1 T \left(\frac{\zeta}{T} + \rho_2\right)^2 + c_2 T (1 - \rho_2)^2 = 0.65\zeta.$$

ACKNOWLEDGEMENT

The author wishes to express his appreciation to the referees for many helpful suggestions.

APPENDIX

In section 2 we made an assumption that $\rho_1 + \rho_2 < 1$, which implies that the server is capable of handling transmitting load. We now prove this contention and show that there exists policies which require that the server be switched from side X to side Y only if the backlog of messages at side X is zero. We call this class to policies C_0. We will also show that there exists policies that belong to C_0 and are stationary.

Let

(i) $T_{\min} = \dfrac{\zeta}{1 - \rho_1 - \rho_2}$, where $\zeta = \zeta_{12} + \zeta_{21}$ and

(ii) the initial backlog of messages at two sides is zero.

THEOREM A–1: Under the assumptions of section 2 above there exists policies that are type C_0.

PROOF: We prove this theorem by a constructive proof.

Let A_i be the time spent by the server in relaying messages from side X to side Y for the ith cycle. Similarly, let B_i be the time spent by the server relaying messages from side Y to side X for the ith cycle.

Let the server start by serving at side X for A_1 units of time. Then switch the server and let it relay messages from side Y for B_1 units of time, where

$$B_1 = \frac{\rho_2}{1-\rho_2}(A_1 + \zeta_{12}).$$

The backlog of messages at side Y at the end of B_1 units of time

$$= \lambda_2(A_1 + \zeta_{12}) - (\mu_2 - \lambda_2)B_1$$

$$= \lambda_2(A_1 + \zeta_{12}) - \lambda_2(A_1 + \zeta_{12}) = 0.$$

Switch the server to side X and let it relay messages at side X for A_2 units of time, where

$$A_2 = \frac{\rho_1}{1-\rho_1}(B_1 + \zeta_{12} + \zeta_{21}).$$

The backlog at the end of A_2 units of time at side X

$$= \lambda_1(\zeta_{12} + B_1 + \zeta_{21}) - (\mu_1 - \lambda_1)A_1 = 0.$$

Now in general, by letting the server serve at side X during ith cycle for A_i units of time where

$$A_i = \frac{\rho_1}{1-\rho_1}(B_{i-1} + \zeta) \qquad \text{for } i = 2, 3, \ldots .$$

We will ensure that the backlog of messages at side X is reduced to zero in each cycle. Similarly the server can serve at side Y for B_i units of time during the ith cycle where

$$B_i = \frac{\rho_2}{1-\rho_2}(A_{i-1} + \zeta) \qquad \text{for } i = 2, 3, \ldots .$$

We have shown above that there exists policies of type C_0.

THEOREM A–2: Under the assumptions of section 2, there exists policies that belong to C_0 and are stationary.

PROOF: In Theorem A–1 we proved that there exists policies of type C_0. We now show that there exists stationary policies among C_0:

Let A_i and B_i be as defined in Theorem A–1.

Let $A_1 = \zeta_{21} + \dfrac{\rho_1\zeta}{1-\rho_1-\rho_2}$.

Let $B_1, B_2, \ldots$, and $A_2, A_3, \ldots$, be as defined in Theorem A–1, then

$$B_1 = \frac{\rho_2}{1-\rho_2}(A_1 + \zeta_{12})$$

$$= \frac{\rho_2}{1-\rho_2}\left(\zeta_{21} + \zeta_{12} + \frac{\rho_1\zeta}{1-\rho_1-\rho_2}\right)$$

$$= \frac{\rho_1}{1-\rho_2}(1-\rho_1-\rho_2+\rho_1)\frac{\zeta}{1-\rho_1-\rho_2} = \frac{\rho_2\zeta}{1-\rho_1-\rho_2}.$$

$$A_2 = \frac{\rho_1}{1-\rho_1}\left(\frac{\rho_2\zeta}{1-\rho_1-\rho_2} + \zeta\right)$$

$$= \frac{\rho_1\zeta}{1-\rho_1-\rho_2}.$$

By a similar calculation we can show that

$$A_i = \frac{\rho_1\zeta}{1-\rho_1-\rho_2} \qquad \text{for } i = 2, 3, \ldots$$

and

$$B_i = \frac{\rho_2\zeta}{1-\rho_1-\rho_2} \qquad \text{for } i = 1, 2, 3, \ldots.$$

This implies that the server spends equal amounts of time at each side during each cycle and hence a stationary policy exists.

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PRODUCTIVITY ESTIMATES OF THE STRATEGIC AIRLIFT SYSTEM BY THE USE OF SIMULATION

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ABSTRACT

Although the strategic airlift system is under continuous analysis, C–5A problems provided impetus to analyze the airlift system productivity function by using a large-scale simulation model. Development of the simulation model (Simulation of Airlift Resources—SOAR) was initiated by the Office of Secretary of Defense (Systems Analysis) in 1966. SOAR had barely become operational in time for the study in November 1968.

Since limited verification and validation tests had been performed on the simulation model, the design of experiments was of critical importance. The experimental design had to be flexible enough to salvage the maximum amount of information possible upon the discovery of either a verification or validation error. In addition, the experimental design was required to accommodate the estimation of a large number of possibly changing independent variables.

The experimental design developed for the analysis was full factorial design sets for a finite number of factors. Initial analysis began with aggregated sets of factors at two levels, and information gained from experiment execution was used to parse the sets. The process was sequential and parsing continued until the major explanatory independent variables were identified or enough information was obtained to eliminate the factor from further direct analysis. This design permitted the overlapping of simulation runs to fill out the factorial design sets.

In addition to estimating the airlift productivity function, several other findings are reported which tended to disprove previous assumptions about the nature of the strategic airlift system.

PRODUCTIVITY ESTIMATES OF THE STRATEGIC AIRLIFT SYSTEM USING SIMULATION

Computer simulation was used in a recent Department of Defense study designed to evaluate the impact of strategic airlift system characteristics on the productivity of the military airlift system.* Although the system is under continuous analysis, budgetary pressures and adverse reaction to the C–5A cost increases prompted the formation of a joint OSD/Air Force study group to evaluate a proposal

* The Mobility Forces Division of Systems Analysis in the Office of the Secretary of Defense performed the analysis with Headquarters, United States Air Force.

The simulation analysis of the strategic airlift system is one component of an optimization/simulation framework for the analysis of strategic mobility. For a description of the framework see R. L. Nolan and M. G. Sovereign, "A Recursive Optimization and Simulation Approach to Analysis with an Application to Transportation Systems," Management Science (August 1972). For a description of the simulation analysis of the strategic sealift component, see R. L. Nolan and M. G. Sovereign, "Simulation of the Strategic Sealift System," *Proceedings of 1971 Summer Computer Simulation Conference*, pp. 1184–1191.

to reduce procurement of the controversial Lockheed transport aircraft. The hypothesis had been made that with a marked increase in the daily utilization rate (i.e., flying hours per day) of aircraft in the transport force, wartime deployment capability could be maintained at the required level with significantly fewer C–5As. A controversy developed over the feasibility of operating the military aircraft force at the proposed higher utilization rate under conditions of a wartime rapid deployment. Of particular concern was the uncertainty over the potential contribution to higher utilization rates of increases in airlift support resources. Previous tests and analytical efforts to determine feasible utilization rates were inconclusive in resolving the question of increased resources or in supporting or rejecting the hypothesis.

A useful analysis of utilization rates, and its derivative productivity, must consider not only the factors affecting utilization rates, but factor interactions and combined impact on the airlift system. Because of the number of variables and the complexity of the system, simulation emerged as the most appropriate method of analysis. In 1966 the Office of Assistant Secretary of Defense (Systems Analysis) had initiated the development of several computer models to aid strategic mobility planning and analysis. One of these models was a Simulation of Airlift Resources (SOAR), a computer simulation of intertheather airlift operations under conditions of rapid deployment which had been designed and developed within the Office of Joint Chiefs of Staff (SASM).†

The complexity of the system and the decision to use simulation posed two different problems in the design of the analysis:

1. Model verification and validation.
2. Experimental design.

The proper resolution of both problems is key to developing useful conclusions. Complete verification of a large simulation model such as SOAR is almost impossible. Validation in the sense of testing the "goodness of fit" of the model with real world data was infeasible for the following reasons:

1. SOAR is one of the largest computer simulations designed.
2. The strategic airlift system has never been exercised in an intertheater rapid deployment.
3. Data from previous exercises of the strategic airlift system are (a) incomplete, and (b) obsolete.
4. Field test data on parts of the system are difficult to interpret because they were not collected for the purposes at hand, as well as the fact that environmental conditions are rarely controlled in such tests.

The task of verification and validation was redefined within the constraints outlined. The objective of the "verification and validation" process is to develop a "threshold of confidence," which can be conveyed to decisionmakers, that the model is a sufficient abstraction of reality to be used as a basis for decisions. A number of techniques are available for building this level of confidence. One of the most important was the involvement of operations managers in the design of the model and interpretation of model-generated results. In addition, statistical analysis of the model results provided a measure of consistency and inherent variability in the model itself.*

† Office of the Joint Chiefs of Staff (Special Assistant for Strategic Mobility).

* Naylor and Finger together with a critique by McKenney state the validation problem and summarize the approaches to it. See Thomas H. Naylor and J. M. Finger, "Verification of Computer Simulation Models," *Management Science* (October 1967), and James L. McKenney, "Critique of Verification of Computer Simulation Models" in the same issue.

SOAR Model

SOAR is programmed by using a derivative of the GASP simulation language, which is, in turn, a derivative of FORTRAN. It can simulate up to 1,000 aircraft, 100 airbases, and 1,500 air crews. Approximately 6 to 7 minutes of computer (IBM 360/65) time are required to simulate the activities of 1 day.

In SOAR the important categories of variables in the strategic airlift system are represented. Air traffic density is a function of the number of mission aircraft (C–141 and C–5A) and the competing aircraft (Civil Reserve Air Fleet (CRAF), tactical airlift, fighters, reconnaissance). Weather effect is simulated by specifying the effect of seasonal enroute winds on leg flying time, fuel requirements and allowable cabin load. In addition, hourly weather observations for each base in the simulation can close a base when conditions of ceiling, visibility, or crosswind dictate. All base resources, comprising a third category of resources, are treated in detail. This category includes the personnel and equipment for cargo handling, maintenance, and refueling, aircrews, parking space, and airfield acceptance data. Airfield acceptance data specify base IFR (Instrument Flying Rules) minima and the number of aircraft per hour which can be managed by approach and landing controllers. A fourth category of variables addressed by the computer model includes those aircraft characteristics (e.g., speed, fuel consumption rates, parking "shadow") which impact system performance. Other system characteristics which may have an effect such as crew staging policies, routing policies, scheduled maintenance policies, and warning time are addressed both in problem formulation and in ordering of input data.

Each aircraft to be simulated is located at a specific base and given a starting time. The cargo to be moved is identified by weight and its location at bases. The effect of cargo density is described by a cumulative frequency distribution which essentially describes the probability of a C–5A or a C–141 being loaded at a given weight. Levels of maintenance, refueling, cargo handling, and aircrew capability are specified for each base. With the scenario and system defined, initialization is begun.

During initialization, the input data are checked for internal consistency and completeness, and the random number generator is used to establish the initial values of those aircraft, aircrew and resource attributes which are stochastic in nature. Following these actions, the simulation is ready to begin.

In SOAR, there are four types of bases (home, onload, enroute, and offload), and at each base are located resources consistent with its function. For instance, at onload and offload bases there can be found aerial port crews needed to process cargo; at home bases, a large pool of aircrews and the facilities necessary for all maintenance short of depot-level; at all bases, of course, are parking spaces, transient maintenance and fuel. The simulation begins with each aircraft at its home base. There the aircraft is checked to ascertain that it has no maintenance problems, an aircrew is ready, and adequate fuel is aboard. If weather permits, the aircraft departs for the airfield that has the highest priority cargo awaiting pick-up. On the aircraft's arrival at the onload base, the model selects the route to be taken from this onload base to the cargo destination (offload base). The route selection is based on factors, such as weight of the cargo, fuel requirements, weather, and the availability of resources at the bases in the several candidate route networks. With the route selected and the cargo now placed aboard, a check is made to assure that the aircraft is still operationally ready in terms of maintenance, aircrew, and fuel. Deficiencies are corrected, and the aircraft departs on its next leg, weather permitting.

The procedure is repeated as the aircraft moves through its enroute bases and arrives at the delivery point. After cargo is offloaded, the model determines which onload base has an appropriate load of cargo to be picked up by this aircraft. Again necessary maintenance and fuel servicing is performed, a rested aircrew replaces the one aboard if required, a weather check is performed and the aircraft departs for its next onload base to recommence the cycle.

As an aircraft moves through the airlift system, from home to onload through a series of enroute bases and finally to offload, it makes demands on the resources available at these bases. Where demands exceed the capability of the bases to meet them, queues form adding to ground time reflecting the penalties resulting from system saturation. Some of the demands are deterministic; aircrews are replaced if duty time will exceed 18 hours, fuel requirements are based on flying time and fuel consumption rates, and the parking area used is the result of "parking shadow" of the aircraft itself.

Other demands are stochastic: maintenance failures are generated randomly from a distribution as a maintenance man-hours required to repair the failure, the routes themselves are selected by the random process which determines the weight of the cargo loaded on each aircraft. Still others are a combination in which a random number generation selects an entry value in a table which then is processed in a deterministic manner.

However arrived at, it is these demands and the ability of supporting systems to satisfy them which are key elements of the airlift simulation. Since only the lack of one or more of the essential resources will hold aircraft on the ground, SOAR is well designed for the evaluation of the impact of changing levels of system resources and demands on aircraft utilization rates.

Design of the Experiments

SOAR is typical of the large-scale simulation model. It consists of many instructions, complex logic, and is expensive to run. Eventually, the model grows beyond the ability of the analyst to comprehend all of the interactions between the components. Hunter and Naylor* suggest the use of a two-stage experimental design for analysis on the large scale simulation model. The first stage can be termed exploratory. The exploratory stage is intended to discover the major contributing independent variables to the response, or dependent variable. Techniques such as factorial analysis are especially useful along with other tools such as regression analysis, analysis of variance, and linear mathematical models. The exploratory stage is followed by an optimization stage which is intended to optimize the response variable, or at least improve it, by adjusting the controllable independent variables.

The experimental design reported in this paper focuses on the exploratory stage. The response variable is aircraft utilization and aircraft productivity, each defined for the C-5A and C-141. In general the response variables (Y) are a function of the factors (X_i),† that is

$$Y = f(X_1, X_2, \ldots, X_{23}).$$

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* J. S. Hunter and Thomas H. Naylor, "Experimental Design," in Thomas H. Naylor, editor, *The Design of Computer Simulation Experiments* (Durham, North Carolina: Duke University Press, 1969), pp. 39–58. Much has been written in general on experimental design techniques. See for example, D. R. Cox, *Planning of Experiments* (New York: John Wiley and Sons, 1961), W. Mendenhall, *Introduction to Linear Models and the Design and Analysis of Experiments* (Belmont, California: Wadsworth Publishing Company, 1968), and Ken Chen Peng, *The Design Analysis of Experiments* (Reading, Mass.: Addison, Wesley, 1967). Only relatively recently have the design of experiments for computer simulation experiments been discussed in the literature. The special problems are focused around the expense of making large-scale computer simulation experiments and the model verification/validation problem.

† Since the analyst may only set levels of controlled variables, the response is characterized as a function of the controlled variables.

where the X_i are the variables listed in Table 1. For any given computer run the variables take on specific values x_i:

$$X_1 = x_1, X_2 = x_2, \ldots, X_{23} = x_{23}.$$

TABLE 1. *SOAR Model Variables*

Variable number	Variable description
1	Mission aircraft
2	Competing aircraft—other tactical aircraft, fighters, etc.
3	Personnel for cargo handling
4	Equipment for cargo handling
5	Maintenance personnel
6	Maintenance equipment
7	Refueling personnel
8	Refueling equipment
9	Air crews
10	Aircraft parking space at a base
11	Airfield capacity
12	Weather data—including the effects of seasonal enroute winds, fuel requirements, allowable cabin load
13	Hourly weather data about the conditions at a particular base
14	Aircraft speed
15	Aircraft fuel consumption
16	Aircraft parking-landing requirements
17	Aircraft maintenance data
18	Crew staging policies at bases
19	Routes and routing policies
20	Scheduled maintenance policy
21	System alert warning time
22	Airbase locations and descriptions
23	Other miscellaneous data

These factors may be considered the independent variables. Indeed, the factors may not be truly independent; their values may be altogether constrained by some other limiting function (i.e., total resources, or an annual budget). Nevertheless in the exploratory stage, the analyst usually accepts the independency assumption and then challenges it with the benefit of more information in the optimization stage. In the optimization stage, appropriate constraints can be employed for searching to define explicitly the experimental region.

In the initial exploratory stage with a model like SOAR, the number of computer runs can cost a prohibitive amount. An experimental design is required that will provide information on all the variables of main interest, but can be executed to unfold successively more information about the more significant variables. The factorial experiment can be used to do this.

The first step in the design of the SOAR experiments was to eliminate from consideration variables of indirect interest, or variables that could be analyzed by use of other means. "System alert time" and "miscellaneous data" were thereby eliminated leaving 21 variables to be considered. The second step was to decide on the levels to be assigned to the variables. For exploratory purposes, each variable was assigned a base or normative level and an augmented level except for the aircrew

variable which was assigned a base level and a decremented level. Each combination of variable levels produced a design point. A full factorial experiment requires computer runs for each design point. Enough information is provided to determine the effects of changing the level of each variable and the effects of variable interactions. However, the information advantage in the SOAR analysis of the full factorial was clearly offset, and infeasible, by the cost of making the computer runs to satisfy the full factorial design. The number of runs included in a full factorial experiment with 21 variables each at two levels is 2^{21}, or 2,097,152.

Two alternatives to the full factorial experiment were examined: fractional factorial, and factor aggregation. The fractional factorial experiment is to make computer runs on a subset of the runs required for a full factorial. The analyst selects a sequence of runs with the factors at various base and augmented levels. Although the problem of factor choice during an experiment is well documented in the literature,* it poses a difficult problem for the analyst at the outset. Thus, it was decided to employ a factor aggregation method.

The factor aggregation method is to group similar variables which are then all set at base levels or augmented levels during any one run. A full factorial experiment is run using these variables groupings as if they were variables themselves. In the SOAR analysis, variables were initially grouped into three categories and full factorial experiments were satisfied. Figure 1 illustrates the general design set. The zero (0) entry indicates the base level and a (1) entry indicates the changed level.

In Figure 2, the same design is shown with the actual variables and levels of a specific case substituted for the general notation. Weather is Variable One, with winter weather representing the (0) entry and summer weather the (1) entry. Variable Two is now All Resources, with (0) being the "base case" and (1) the changed (augmented) case. Variable Three is Air Traffic. In the base case (0 level),

Encounter	Variable One	Variable Two	Variable Three
1	0	0	0
2	1	0	0
3	0	1	0
4	1	1	0
5	0	0	1
6	1	0	1
7	0	1	1
8	1	1	1

FIGURE 1. General Design Set

Encounter	Weather	All Resources	Traffic
1	Winter	Base level	MAC + CRAF
2	Summer	Base level	MAC + CRAF
3	Winter	Augment 50%	MAC + CRAF
4	Summer	Augment 50%	MAC + CRAF
5	Winter	Base level	MAC + CRAF + TAC
6	Summer	Base level	MAC + CRAF + TAC
7	Winter	Augment 50%	MAC + CRAF + TAC
8	Summer	Augment 50%	MAC + CRAF + TAC

FIGURE 2. Typical Experimental Design Set

*John L. Overholt, "Factor Selection," in Thomas H. Naylor, editor, *op. cit.*, pp. 59–79.

only CRAF airframes compete with mission aircraft for resources; in the augmented case (1 level), TAC (Tactical Air Command) activity is added to the MAC (Military Air Command) and CRAF traffic.

The impact of the categories, or independent variables, on the dependent variable, either aircraft utilization rate or productivity, was assessed by applying a standard analysis of variance technique. Three factor analysis of variance with interactions shows the statistical significance of the change in resource level and the effect of factor interaction.

The statistical analysis of the 2^3 factorial experiment* is used to determine whether further experimental analysis of a category is warranted. If a category is statistically significant, a second experimental design is then formulated to test that category again with other categories. Alternatively, the category may be parsed into two or more less aggregated variables in order to discover the actual variables that are contributing to the dependent variable. On the other hand, statistically insignificant categories may be temporarily dropped from further analysis.

For instance, as it became obvious that the variable All Resources had a significant impact, it was necessary to determine which particular resource types were the most important marginal contributors. Therefore Refueling and Maintenance were grouped in a separate category as were Aircraft Acceptance Rates and Parking Space. These two categories were then run in a subsequent design set with Weather providing the third variable. If the experiments demonstrated that a particular variable had no significant effect, it was not included in future sets. The same procedure was used when it was determined that enough information had been gathered about a particular variable. For example, in the SEAsia case Weather was a variable in 16 runs. In the last six runs (a total of 22 SEAsia encounters were run in the experiments) only winter weather was used since it was deemed that enough information concerning the marginal contribution of seasonal weather changes had been collected.

SOUTHEAST ASIA ANALYSIS

Twenty-two computer simulations were made with the SOAR model for the SEAsia case. In these runs, the objective was to quantify the impact on the airlift system of such variables as weather, base resources, competing traffic, aircrews, and mission aircraft. The impact of changes in the levels of the variables was measured in terms of their effect on utilization rates and productivity. Three different tools of statistical analysis were employed: three factor analysis of variance† was used primarily to test the level of significance of the changes in the dependent variables caused by the factors and interactions being evaluated; paired comparisons were used to identify the marginal contributions of the variables; and regression analysis was used to examine the relationship of utilization rate to productivity.

Figure 3 illustrates the independent variables which were changed during the 22 SEAsia runs.

In the first experiment, the three factors were Weather, Cargo-handling Resources and Other Base Resources; for Weather, the base level was winter; the changed level was summer. For the

*$2^3 = 8$, the number of possible combinations of two levels for each of the three factors being tested. Each SOAR run is equivalent to one observation in the 2^3 factorial experiment.

†The use of three factor analysis of variance is an unconventional use of a familiar statistical technique. Among statisticians, there is a difference of opinion as to whether this technique is appropriate to the purpose to which it was applied. The adequacy of eight observations (SOAR runs) to test the three variables and their interactions is the point in question. The eight observations provide a total of seven degrees of freedom. Of these, one each is used for each variable and interaction. The remaining one degree of freedom is applied to experimental error which also includes the high order interaction. Many statisticians would insist on more than one degree of freedom for experimental error. The point in question is the power of the test, the level of confidence one may place in the results of the test.

Run	V_1	V_2	V_3	V_4	V_5	V_6	V_7
1	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0
3	0	1	1	1	0	0	0
4	1	1	1	1	0	0	0
5	0	0	1	1	0	0	0
6	1	0	1	1	0	0	0
7	0	1	0	0	0	0	0
8	1	1	0	0	0	0	0
9	0	0	1	0	0	0	0
10	1	0	1	0	0	0	0
11	0	0	0	1	0	0	0
12	1	0	0	1	0	0	0
13	0	0	0	0	1	0	0
14	1	0	0	0	1	0	0
15	0	1	1	1	1	0	0
16	1	1	1	1	1	0	0
17	0	0	0	0	1	0	−1
18	0	1	1	1	1	0	−1
19	0	0	0	0	1	1	0
20	0	1	1	1	1	1	0
21	0	0	0	0	1	1	−1
22	0	1	1	1	1	1	−1

FIGURE 3. SEAsia Experiments

two other categories, the base level consisted of those men, equipment, materiel, facilities, or capability assigned to each airbase. In the changed case, the base level of resources was augmented at every airbase by a constant percentage, usually 50 percent. The results of the first experiment are shown in Figure 4 for the C–141 and C–5A.

The presence of the word "none" in Figure 4 indicates that the factor or interaction was not statistically significant and that any impact thought to be associated with it most likely occurred by chance. In fact, the level of significance indicates the probability that the effect being measured occurred by chance. Thus, the smaller the number, the more confident we can feel that the factor being tested actually was associated with the impact that was reflected in utilization rates or productivity. For example, looking at C–5A utilization rate, the 0.08 level of significance indicates a probability of 8 percent that the impact occurred through random effects and a 92 percent probability that it may be attributed to weather. Furthermore, by consulting the Marginal Contribution chart, one sees that the mean difference between summer and winter weather was 0.75 hours (45 min per aircraft day). Other Base Resources also affected utilization rates; the effect was statistically significant at the 0.05 level. Only Cargo-handling seemed to have no significant impact on C–5A rates. The C–5A rates were also relatively sensitive to the interaction of the variables, although there was a good chance that such an impact could have occurred through random effects (25 to 37 percent).

The C–141, on the other hand, was not sensitive to any changes in the factors or interactions. With the single exception of the effect of weather on productivity, there is reason to believe the differences which were registered actually occurred by chance.

In Experiment 1, Cargo-handling was shown to have a negligible impact on both aircraft. Weather and Other Base Resources, however, had an important effect on C–5A utilization rates and a strong indication for C–5A productivity. Therefore Weather was retested in Experiment 2. Other Base Re-

Marginal contributions

Category	C–141		C–5A	
	Utilization rate (hours per day)	Ton-mile productivity per day (000)	Utilization rate (hours per day)	Ton-mile productivity per day (000)
A. Weather	0.11	1.32	0.75	10.05
B. Cargo-handling Resources	0.12	0.42	0.04	−0.57
C. Other Resources	0.19	1.04	0.77	10.98

Levels of significance

Category	C–141		C–5A	
	Ute rate	Productivity	Ute rate	Productivity
A. Weather	0.75	0.50	0.08	0.25
B. Cargo-handling	0.75	None	None	None
C. Other Resources	0.75	0.75	0.05	0.25
AB. Interaction	0.60	0.75	0.25	None
AC. Interaction	None	None	0.37	0.75
BC. Interaction	0.75	0.75	0.25	None

FIGURE 4. Experiment 1

sources was subdivided into two more definitive categories; (1) Refueling and Maintenance, and (2) Airfield Acceptance Capability and Parking. The results of Experiment 2 are shown in Figure 5.

With the possible exception of Airfield Acceptance Capability and Parking, none of the categories in Experiment 2 were shown to be significant at a meaningful level. This evidence appeared to confirm the indications of Experiment 1 that the levels of resources in the base case were too high for the level of demand. As a result it was decided to stress the system by increasing demand.

In Experiment 3, Weather was tested once again. All Base Resources was the second variable tested (it actually was the second test for this category since in Experiment 1, Cargo-handling Resources and Other Base Resources add up to All Base Resources). The third category was Competing Aircraft. In the base case, competing aircraft consisted of commercial carriers (CRAF) only. In the augmented case, the deploying tactical air forces (TAC) were added to CRAF. Results appear in Figure 6.

Once more, the experiment illustrated the relative insensitivity of C–141 to changes in the operating environment. The increased competition for resources probably had such a negligible effect that it was entirely overcome by random effects which produced an *increase* in utilization rates and productivity for the C–141. The highest utilization rate for all 22 SOAR runs was recorded in this set; the C–141s averaged 15.19 hours in the Summer case having augmented resources and the fully augmented level of competing aircraft.

The greatest impact for the categories being tested were found in C–5A productivity. All of the factors and interactions were determined to have a statistically significant impact at levels ranging from 0.005 to 0.10. The addition of competing aircraft resulted in the loss of an average of 1.17 hours and 6,690 ton miles per aircraft per day.

Marginal contributions

Category	C–141		C–5A	
	Utilization rate (hours per day)	Ton-mile productivity per day (000)	Utilization rate (hours per day)	Ton-mile productivity per day (000)
A. Refuel/Maintenance	−0.24	0.08	0.09	1.96
B. Weather	0.08	0.85	0.12	7.38
C. Airfield/Parking	0.29	1.81	0.53	8.97

Levels of significance

Category	C–141		C–5A	
	Ute rate	Productivity	Ute rate	Productivity
A. Refuel/Maintenance	0.50	None	None	0.75
B. Weather	None	0.75	0.50	0.35
C. Airfield/Parking	0.50	0.40	0.40	0.25
AB. Interaction	None	None	None	None
AC. Interaction	0.50	None	0.75	0.75
BC. Interaction	0.75	None	0.75	0.40

FIGURE 5. Experiment 2

Marginal contributions

Category	C–141		C–5A	
	Utilization rate (hours per day)	Ton-mile productivity per day (000)	Utilization rate (hours per day)	Ton-mile productivity per day (000)
A. All Resources	0.62	2.79	1.36	16.39
B. Weather	0.12	1.26	0.56	7.41
C. Competing Aircraft	0.28	1.37	−1.17	−6.69

Levels of significance

Category	C–141		C–5A	
	Ute rate	Productivity	Ute rate	Productivity
A. All Resources	0.25	0.25	0.40	0.005
B. Weather	0.75	0.50	0.75	0.05
C. Competing Aircraft	0.50	0.50	0.50	0.10
AB. Interaction	None	None	0.50	0.10
AC. Interaction	0.50	0.50	0.75	0.05
BC. Interaction	0.50	0.50	None	0.05

FIGURE 6. Experiment 3

In Experiment 4, Weather was dropped from further consideration because sufficient data had been accumulated: it accounted for an average marginal increase in productivity for "C–5As" of 6,333 ton miles per day and 1,771 ton miles per day for C–141s. In order to observe the system under maximum stress, Mission Aircraft was evaluated as a variable. In the base case, the number of C–5As and C–141s were 67 and 157, respectively; in the augmented case, the C–5As were increased to 96 and the C–141s to 224. The second category tested was All Base Resources. Flight Crews constituted the third category. In the base case, 1,464 crews were used; in the changed case (decremented, this time) the number was 1,246. The results of Experiment 4 are shown in Figure 7.

Marginal contributions

Category	C–141		C–5A	
	Utilization rate (hours per day)	Ton-mile productivity per day (000)	Utilization rate (hours per day)	Ton-mile productivity per day (000)
A. All Resources	0.87	3.70	2.20	23.64
B. Flight Crews	−0.16	−0.64	−0.18	−1.56
C. Mission Aircraft	−0.89	−3.94	−0.84	−21.23

Levels of significance

Category	C–141		C–5A	
	Ute rate	Productivity	Ute rate	Productivity
A. All Resources	0.10	0.20	0.15	0.20
B. Flight Crews	0.50	0.75	0.75	None
C. Mission Aircraft	0.15	0.20	0.30	0.20
AB. Interaction	0.75	0.75	0.35	0.75
AC. Interaction	None	None	None	0.50
BC. Interaction	0.50	None	None	None

FIGURE 7. Experiment 4

As expected, the increased demand generated by the additional mission aircraft caused the category "All Base Resources" to register a more impressive impact. It was statistically significant at levels ranging from 0.10 to 0.20. Changes in the levels of "All Base Resources" accounted for a variation of 2.20 hours and 23,640 ton miles per day for each C–5A. Even the relatively insensitive C–141 varied an average of 0.87 hours and 3,700 ton miles per day in response to changes in "All Base Resources."

The addition of more mission aircraft caused a decline in utilization rates and productivity per aircraft. This is a further illustration of the effect of queueing brought on by increased competition. It is also an indication that airlift forces productivity is not directly proportional to the number of aircraft applied. That is: the productivity of one C–141 may be 60,000 ton miles per day, but the productivity of 100 C–141s is probably less than 6 million ton miles (i.e., $100 \times 6{,}000$).

EUROPE/AFRICA ANALYSIS

Twenty computer simulations were made with the SOAR model for the simultaneous deployments to Europe and Africa. Because these runs were made after the SEAsia runs had been completed, they were designed with the benefit of more experience with the computer model and a better understanding of system behavior. Thus, it was possible initially to focus these experiments on specific resources. In addition, after the first design set, augmentation was limited to those resources or capabilities which could reasonably be expected to be dramatically improved. In the dual deployment, 15 C–5A and 105 C–141 aircraft were allocated to the African deployment; the remaining 52 C–5As and 42 C–141s were assigned to European requirements. Figure 8, a summary chart of the 20 SOAR encounters, shows the variables tested in the Europe/Africa runs.

Run	V_1	V_2	V_3	V_4	V_5	V_6	V_7
1	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0
3	0	1	1	1	1	1	0
4	1	1	1	1	1	1	0
5	0	0	0	0	0	0	1
6	1	0	0	0	0	0	1
7	0	1	1	1	1	1	1
8	1	1	1	1	1	1	1
9	0	1	1	0	0	0	0
10	1	1	1	0	0	0	0
11	0	0	0	0	1	0	0
12	1	0	0	0	1	0	0
13	0	1	1	0	1	0	0
14	1	1	1	0	1	0	0
15	0	0	0	1	0	0	0
16	0	1	0	0	0	0	0
17	0	1	0	1	0	0	0
18	0	0	1	0	0	0	0
19	0	0	1	1	0	0	0
20	0	1	1	1	0	0	0

V_1: Weather; 0 = Winter, 1 = Summer.
V_2: Refueling; 0 = Base level, 1 = augmented level.
V_3: Maintenance; 0 = Base level, 1 = augmented level.
V_4: Cargo-Handling; 0 = Base level, 1 = augmented level.
V_5: Parking; 0 = Base level, 1 = augmented level.
V_6: Other Resources; 0 = Base level, 1 = augmented level.
V_7: Flight Crews; 0 = 963 crews, 1 = 1187 crews.

FIGURE 8. Europe/Africa Case

As before, eight SOAR encounters were used to provide the observations for the 2^3 factorial design sets. Three experiments were executed. In Experiment 1, the variables tested were: Weather, Flight Crews, and All Airbase Resources. Once again, the base Weather level consisted of winter wind factors and airfield observations; summer provided the changed level. For Flight Crews, the programmed 4.42 crew ratio was used for the base level; the changed (augmented level) was a 5.42 crew ratio. For the base level for All Airbase Resources, the study group's best estimate of men, equipment, materiel, and capability available at the system's airbases was used. For the augmented level, these resources

were increased system-wide by a flat percentage, usually approaching 50. The results of Experiment 1 are shown in Figure 9.

Marginal contributions

Category	C-141		C-5A	
	Utilization rate (hours per day)	Ton-mile productivity per day (000)	Utilization rate (hours per day)	Ton-mile productivity per day (000)
A. Crews	0.01	− 0.25	− 0.30	− 0.31
B. Weather	− 0.01	0.59	− 0.15	− 1.1
C. All Airbase Resources	3.27	13.49	3.59	48.69

Levels of significance

Category	C-141		C-5A	
	Ute rate	Productivity	Ute rate	Productivity
A. Crews	None	None	0.10	None
B. Weather	None	0.75	0.20	None
C. All Airbase Resources	0.05	0.05	0.005	0.05
AB. Interaction	None	None	0.05	0.50
AC. Interaction	0.75	None	0.10	None
BC. Interaction	0.75	None	0.05	None

FIGURE 9. Experiment 1

The results of the paired data tests used to compute marginal analysis indicate that seasonal changes in weather had little effect on utilization rates. Since one might anticipate a significant effect from weather in northern Europe, the suspicion arises that the weather enroute to Africa somehow dampened or washed out the effects of European weather. This hypothesis was supported by a separate analysis of weather effects by geographical area.

The negative effects of the addition of one extra flight crew per aircraft may be attributed to random effects. Apparently the base level crew ratio was sufficient to support airlift activity at the levels tested; the additional crews were neither required nor used. Therefore, the net effect was that of additional runs of the base cases with random effects providing a negligible variation in results. Augmentation of All Airbase Resources resulted in a dramatic increase in both utilization rates and productivity. The impact was significant at levels ranging from 0.05 to 0.005. The important increases resulting from the augmentation of All Base Resources indicated that a more detailed examination of this aggregated category might be productive. Accordingly Experiment 2 was designed to test at two levels the following variables: Base Parking Space, Refueling and Maintenance (combined), and Weather. All other variables were set at base levels. Results of Experiment 2 appear in Figure 10.

The evidence shown in Marginal Contributions for Experiment 2 reinforces the indications from Experiment 1. The important contribution made by All Airbase Resources in Experiment 1 is essentially duplicated by the combined contribution of Refueling/Maintenance and Parking. The aggregate category, Refueling/Maintenance, is shown to be an important marginal contributor; the average difference between the base level productivities and the augmented level of productivities is shown to be

Marginal contributions

Category	C–141		C–5A	
	Utilization rate (hours per day)	Ton-mile productivity per day (000)	Utilization rate (hours per day)	Ton-mile productivity per day (000)
A. Refuel/Maintenance	2.88	11.59	2.73	33.88
B. Weather	0.05	0.49	0.17	1.61
C. Parking	0.17	1.44	0.62	7.87

Levels of significance

Category	C–141		C–5A	
	Ute rate	Productivity	Ute rate	Productivity
A. Refuel/Maintenance	0.08	0.05	0.08	0.10
B. Weather	0.50	0.75	0.75	0.75
C. Parking	0.75	0.25	0.25	0.35
AB. Interaction	0.60	None	0.50	0.60
AC. Interaction	0.50	0.25	0.75	0.75
BC. Interaction	None	0.75	0.75	0.75

FIGURE 10. Experiment 2

Marginal contributions

Category	C–141		C–5A	
	Utilization rate (hours per day)	Ton-mile productivity per day (000)	Utilization rate (hours per day)	Ton-mile productivity per day (000)
A. Refueling	2.59	10.66	2.42	29.64
B. Cargo-handling	0.06	0.61	0.06	0.00
C. Maintenance	− 0.03	− 0.21	− 0.23	− 8.80

Levels of significance

Category	C–141		C–5A	
	Ute rate	Productivity	Ute rate	Productivity
A. Refueling	0.025	0.01	0.05	0.08
B. Cargo-handling	0.40	0.20	0.75	None
C. Maintenance	None	0.50	0.40	0.50
AB. Interaction	0.40	0.20	0.90	None
AC. Interaction	0.25	0.10	0.75	0.75
BC. Interaction	0.75	0.35	0.75	0.75

FIGURE 11. Experiment 3

11,590 ton miles per aircraft day and 33,880 ton miles for the C–141 and C–5A, respectively. The impact of this change is statistically significant at levels ranging from 0.10 to 0.05. The marginal contribution of additional parking space is slight, indicating that the amount of parking presently available would not be a serious constraint on utilization rates ranging from 8.88 hours to 13.46 hours.

Once again Weather was found to be a negligible factor. Therefore, Weather was not tested in the third and last experiment. Because of its slight impact, Parking was also dropped from further consideration. But Refueling and Maintenance were separated with each providing one variable in Experiment 3. The third variable was Cargo-handling. The results of the third Experiment appear in Figure 11.

In Experiment 3, all experiments were run with winter weather. With the exception of those variables being tested, all resources and capabilities were held to the base levels. The results indicate that Refueling capability is the key variable in the Europe/Africa depolyment. Experiment 1 identified the important and significant contribution played by All Airbase Resources. In Experiment 2, the identity of the critical resource was narrowed down to the aggregate category Refueling and Maintenance. In Experiment 3, it was revealed that Refueling was the single most constraining factor in the system. The next logical step in the analytical process would be to identify those key bases in the network at which refueling is a serious problem. It could well be that the addition of a dozen or so refueling trucks, relatively easy to deploy, could eliminate the choke point and make possible significant increases in airlift effectiveness. Also, a change in the number or scheme for recovery bases may alleviate the problem.

CONCLUSION

The simulation findings were instrumental in arriving at the joint OSD/Air Force determination that while some increase in aircraft utilization rates could be achieved by improvements in system support resources, it would not be feasible to operate the airlift force at the proposed utilization rate during a wartime deployment. The simulation was especially valuable in identifying constraints in the system and in estimating the kinds and levels of resource augmentation required to relieve the constraints.

In addition to the analysis of the effect of individual resources, an examination of several relationships commonly assumed to exist between measures of airlift effectiveness and certain system variables was conducted. Two of the most interesting findings concerned the assumptions of direct proportionality between productivity and utilization rate and between productivity and numbers of aircraft.

Productivity is generally recognized as the key measure of effectiveness because of its obvious, direct impact on mission accomplishment. The relationship between productivity and utilization rates (an interim, easily quantifiable measure), has long been assumed to be one of direct proportionality. Regression analysis was employed to determine the relationship between the two variables. The relationship was shown to be linear within the range of the values tested. The assumption of direct proportionality, however, was shown, by the presence of a constant, to be an incorrect one. The size and sign of the constant were, in turn, found to be a function of aircraft type and deployment area. The regression equations suggest the existence of an appropriate predictive expression for each aircraft type and specific contingency area. The development and use by contingency planners of a family of predictive equations would result in more accurate estimates of productivity, closure dates, and airframe requirements.

Direct proportionality has also been assumed between the productivity of an airlift force and the numbers of aircraft in that force. Thus, if the daily productivity of one C–141 is 54 kiloton miles, then the productivity of 10 is 540 kiloton miles and the productivity of 100 is 5,400 kiloton miles. The assumptions underlying such calculations are that marginal increases per aircraft are uniform regardless of the size of the airlift force. As expected, this assumption has been found to be incorrect. As aircraft are added to the base programmed level, total productivity rises, but per aircraft productivity declines. The important point here is the provision of a method to quantify an alternative assumption to the direct proportionality assumption.

With the capability to cope with extremely complex aspects of dynamic systems that are clearly beyond the capability of most mathematical models, simulation has been demonstrated to be a valuable technique for airlift system analysis. The key to controlling a large simulation analysis, such as the SOAR analysis, resides in a carefully constructed experimental design which provides information in a structured manner and which, at the same time, permits the flexibility for the analyst and decision-maker to test new hypotheses developed from additional information that comes available during the analysis. A useful technique developed for this analysis is to design experiment sets for a finite number of factors. Initial analysis begins with aggregated sets of factors, and information gained from experiment execution is used to parse sets and ultimately identify major explanatory independent variables. Moving from aggregate to the individual variables also permits overlap of computer runs for experiments. Thus, the technique facilitates obtaining the maximum information from the minimum number of computer runs.

OPTIMAL LOCATION OF PUBLIC FACILITIES*

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ABSTRACT

Location of both public and private facilities has become an important consideration in today's society. Progress in solution of location problems has been impeded by difficulty of the fixed charge problem and the lack of an efficient algorithm for large problems.

In this paper a method is developed for solving large-scale public location problems. An implicit enumeration scheme with an imbedded transportation algorithm forms the basis of the solution technique.

INTRODUCTION

The location of industrial plants, hospitals, fire stations, parks, electric power plants, etc., has become an important consideration in today's society. Increased demands for public services, low budgets, increased costs, and more competition have stressed the importance of the optimal location of both public and private facilities. This interest has, in turn, increased the importance of research on the fixed charge aspect of location problems and on finding an efficient algorithm for large location problems. It is the purpose of this paper to develop an efficient algorithm for a particular form of the public facility location problem.

FORMULATION OF LOCATION MODEL

The objective function of public facility location problems, as was pointed out by Marks et al. [9], is not always easily defined in quantifiable or commensurate terms. In the location of these facilities, therefore, the objective of maximizing the benefits to society cannot be easily expressed in dollar values. Therefore, an alternate procedure of using a surrogate objective function, as suggested by ReVelle et al. [12] and Teitz [14], was used in the model formation of this paper. Surrogates with some rational basis which have been used successfully are minimum time or average distance traveled by the users of the facility, maximum distance or time between any two facilities, and maximum utilization of the facility.

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** This work was completed when the authors were at The John Hopkins University.

The model formulated in this paper considers the construction of a number of facilities with the objective of maximizing the number of users served by all of the facilities. Potential facility sites must be identified before the model is used. At each site it is possible to build one of several different types of facilities. Facilities may be classified into types by their capacities, by the degree of service they provide, or by their "drawing power" for users.

In order to express the relationship between facility location and number of users, the area to be served is divided into discrete population centers. For each population center the number of users to be contributed to a facility may be expressed as some function of characteristics of the population (economic level, population density, etc.), travel distance to the facility, and facility type. Estimation of the parameters of this function may be performed by a statistical regression [3].

In many facility location problems there is an interaction between two neighboring sites. If facilities are constructed at both sites, they will compete for users. This interaction makes the number of users contributed to a site from any population center a function of the number of neighboring facilities. There are serious difficulties in modelling this kind of interaction. Fortunately, in many public facility problems, where there is no control over population being served, it is likely that each person will use the nearest facility, and it can be assumed that each population center will send its full contribution of users to the nearest facility. The size of the contribution, of course, remains a function of distance to the facility and other variables.

Budget limitations have been handled in several different ways in various location models. In some cases [4, 13] the facility costs are made an explicit part of the objective function to be minimized. In other cases, a constraint on total cost is included in the formulation. A third approach [10, 11, 15] limits the number of facilities to be constructed. This approach will be used in this paper, with specified limits on the number of each type of facility to be constructed.

The resulting location model is

$$\text{Maximize facility use} = \sum_{k=1}^{p} \sum_{j=1}^{n} \sum_{i=1}^{m} S_{ij}^{k} X_{ij}^{k}$$

subject to:

(1) $$\sum_{k=1}^{p} \sum_{j=1}^{n} X_{ij}^{k} = 1 \qquad i = 1, 2, 3, \ldots, m$$

(2) $$\sum_{k=1}^{p} Y_{j}^{k} \leq 1 \qquad j = 1, 2, 3, \ldots, n$$

(3) $$\sum_{j=1}^{n} Y_{j}^{k} = M^{k} \qquad k = 1, 2, 3, \ldots, p$$

(4) $$X_{ij}^{k} \leq Y_{j}^{k} \qquad i = 1, 2, 3, \ldots, m \qquad j = 1, 2, 3, \ldots, n \qquad k = 1, 2, 3, \ldots, p$$

(5) $$X_{ij}^{k}, Y_{j}^{k} = \{0, 1\} \qquad i = 1, 2, \ldots, m \qquad j = 1, 2, \ldots, n \qquad k = 1, 2, \ldots, p$$

where

i = population index
j = facility index
k = facility type index
m = number of population nodes (centers)
M^k = number of facilities of type k to be constructed
p = number of facility types
n = number of facility locations
S_{ij}^k = number of individuals expected to go from population center i to facility of type k at location j, if j is the facility which serves center i
$X_{ij}^k = 1$ if population node i assigns to type k facility at location j
$= 0$ otherwise
$Y_j^k = 1$ if there is a facility of type k at location j
$= 0$ otherwise.

The first constraint requires that each population center assign to one and only one facility of one specific type, and the next constraint limits the number of types of facilities that can be built at a specified location to be less than or equal to one. Since the facility location variables are zero or one, if the facility location is not used the left-hand side will be equal to zero. The third constraint is the budget constraint. It requires the number of facilities of a given type to be equal to the "budget" for that type. The value of this "budget" is exogenous to the model. The last constraint can be considered as the tie constraint in that it connects the assignment variables to the facility variables. It states that in order for an assignment to be made to a particular facility, that facility must exist ($Y_j^k = 1$).

Solution Approach

The model of the foregoing section is a zero-one integer linear program. The technique selected for solution of the problem is a modified form of implicit enumeration [1, 6, 7, 13]. In effect, implicit enumeration is used to select likely sets of facility sites, without consideration of types of facilities to be constructed at each site. A standard Hitchcock transportation algorithm is then used to allocate facility types to the selected sites.

The model is converted to a type-free formulation by the following transformations:

(a) Replace the variables Y_j^k with new variables $Y_j = \sum_{k=1}^{p} Y_j^k$.

(b) Replace the variables X_{ij}^k with new variables $X_{ij} = \sum_{k=1}^{p} X_{ij}^k$.

(c) Replace the constants M^k with new constant $N = \sum_{k=1}^{p} M^k$.

(d) Replace the constants S_{ij}^k with new constants $S_{ij} = \max_k \{S_{ij}^k\}$.

The resulting formulation (called Problem A) is:

$$\text{Maximize facility use} = \sum_{j=1}^{n} \sum_{i=1}^{m} S_{ij} X_{ij},$$

subject to

$$\text{(1a)} \qquad \sum_{j=1}^{n} X_{ij} = 1 \qquad i = 1, 2, 3, \ldots, m$$

$$\text{(3a)} \qquad \sum_{j=1}^{n} Y_j = N$$

$$\text{(4a)} \qquad X_{ij} \leq Y_j \qquad i = 1, 2, 3, \ldots, m; \qquad j = 1, 2, 3, \ldots, n$$

$$\text{(5a)} \qquad X_{ij}, Y_j = \{0, 1\} \qquad i = 1, 2, 3, \ldots, m; \qquad j = 1, 2, 3, \ldots, n.$$

The selection of $S_{ij} = \max_k \{S_{ij}^k\}$ results in an overestimate by Problem A of the facility utilization which can be obtained by any solution.

Problem A may be readily solved by implicit enumeration. However, since a solution of Problem A violates the constraints on facility types, one additional step is added in the implicit enumeration technique. Whenever a new candidate for optimality to Problem A is found, the selected sites are used in a Hitchcock transportation problem to determine optimal allocation of the M^k facilities of each type among the sites. This problem is formulated as follows:

(a) Let $F = \{j'\}$ be the set of sites used in the current solution in the implicit enumeration scheme. These sites become the sinks in the transportation problem.

(b) Let Q be an arbitrarily large constant, and let $C_{kj'} = Q - \sum_{i=1}^{m} S_{ij'}^k$ be the "cost" of assigning a facility of type k to location j'.

The resulting transportation problem (called Problem B) is

$$\text{Minimize "Cost"} = \sum_{k=1}^{p} \sum_{j' \epsilon F} C_{kj'} f_{kj'},$$

subject to

$$\sum_{j' \epsilon F} f_{kj'} \leq M^k \qquad k = 1, 2, \ldots, p$$

$$\sum_{k=1}^{p} f_{kj'} = 1 \qquad \text{for all } j' \epsilon F.$$

This formulation in effect creates a "source" for each facility type, with an amount available equal to the number of facilities of that type to be built. There is a "sink" for each site, with an amount required equal to 1. The "cost" of assigning a facility of type k to site j' is simply the complement of the expected facility use, and is used so that the standard (minimization) transportation form can be utilized. A sketch of a sample problem is shown in Figure 1. The effect of solving this problem is to maximize the total utilization with the selected sites, while observing the constraints on facility types.

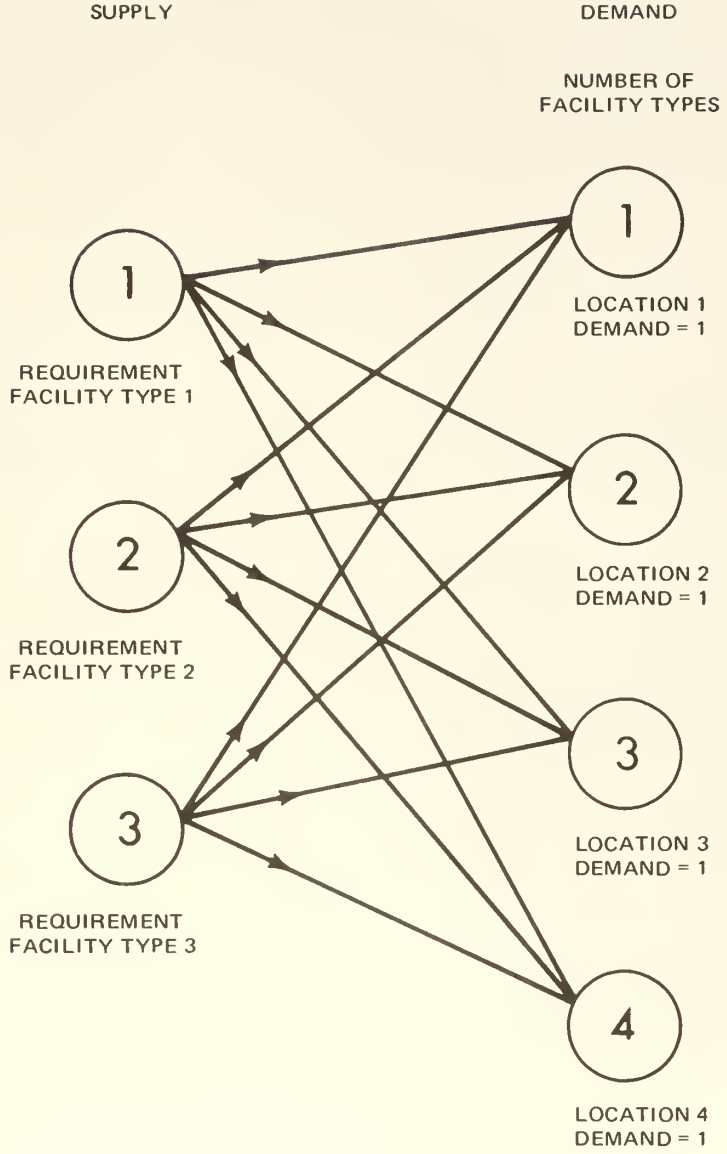

FIGURE 1

The actual utilization may be readily found by taking the complement of the objective function, and will, of course, be less than the overestimate found in Problem A.

The implicit enumeration method is described in terms of "partial solutions" of Problem A. A partial solution consists of a set of variables Y_j which have been (tentatively) set at 1, another set which have been (tentatively) set at 0, and a third set for which no decision has yet been made. Since each population center will assign to its closest facility in any complete solution, no determination can be made of the values of the X_{ij} in a partial solution. As an approximation, it may be assumed that $X_{ij}=Y_j$ for all i and j. This, of course, violates constraint (1a) of Problem A, and results in multiple counting of some population centers.

Suppose such a partial solution has been constructed, and it is desired to determine whether further efforts should be made to complete it. A preliminary check may be made by considering all

of the sites where either $Y_j = 1$ or Y_j is not yet set. If the upper limit on total utilization using the best N of these sites is not greater than that of some previously found complete solution, then the current partial solution is clearly unprofitable to pursue. On the other hand, if this upper bound indicates that the partial solution is promising, a sufficient number of Y_j's may be set to 1 to complete the solution, and the approximate Problem A objective function calculated. If the solution continues to be promising, Problem B may be solved. The Problem B solution gives an exact, feasible solution of the original problem. If the objective function is better than the previous best, the new complete solution becomes the best otherwise a different group of sites is selected to complete the solution and the procedure is followed again.

This approach obviously requires a good bookkeeping scheme for insuring that all possibilities are tried (or partial solutions including them are tried and abandoned) and that identical combinations of sites are not tried more than once. The method used is essentially that of Geoffrion [6].

Computational Experience

This location model was tested for its efficiency on randomly generated data and these results appear in Table 1. In this table the columns are defined as:

m = number of population centers considered
n = number of facility locations
p = number of types considered
required = number of facilities required to be in the optimal solution
iter = number of interations of the enumeration procedure.
t = total execution time in seconds on an IBM 7094.

The results of these runs are consistent with the experience of other implicit enumeration techniques in that the number of iterations depends on the specific problem considered. However, the two-stage technique seems to be an efficient approach to solving large location problems (2).

The authors will make available computer code documentation upon request.

TABLE 1. *Computational Results*

Problem number	m	n	p	required	iter	t(sec)
1	10	10	1	5	6	0.01
2	15	15	1	8	9	0.03
3	20	20	1	8	9	0.05
4	100	6	3	3	4	1.31
5	100	30	3	3	122	227
6	100	30	3	18	19	91
7	100	40	3	3	35	96
8	100	40	3	18	19	116
9	200	5	3	3	5	2.46
10	200	10	3	3	13	17
11	200	10	3	9	10	5.73
12	200	20	3	6	8	30
13	200	20	3	12	13	60
14	200	20	3	18	19	88

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A PROBABILISTIC EVALUATION OF HELICOPTER LIFT CAPABILITY*

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ABSTRACT

This paper describes a technique for the calculation of the probability that a helicopter can lift a specified load, or number of loads with a specified frequency distribution, in a given geographical region. This probability is computed by determining the bivariate altitude-temperature probability distribution for the specified region. The payload capability at any given temperature and altitude is calculated from standard performance equations. By integrating this over the altitude-temperature distribution, it is possible to obtain the probability distribution of payload capability, from which the required probabilities of lifting specific loads can be determined.

INTRODUCTION

The lift capability of a helicopter depends on the ambient altitude and temperature conditions. Therefore, in a specified geographical region, the payload carried varies with the elevation of the point of takeoff and the temperature at the time of takeoff. The probability of being able to lift a specified load at a random point and time in the region thus depends on the altitude-temperature characteristics of the region. The probability distribution of altitude can be constructed from digital terrain data or detailed contour maps. The conditional distribution of temperature, given an altitude, can be determined from weather station data. Thus, a bivariate altitude-temperature distribution can be generated. The payload capability of the helicopter at a given temperature and altitude can be computed with standard performance equations. By integrating this over the altitude-temperature distribution, a probability distribution of payload capability is obtained. If the frequency distribution of the loads to be carried is available, the probability of mission accomplishment can be computed. This probability provides a useful measure with which to compare the effectiveness of competing helicopter designs.

The evaluation process is summarized as follows:

1. Determine the joint probability density function, $f_{A,T}(a, t)$ of the altitude A and temperature T in the region under consideration.

2. Determine the probability distribution function $F_Y(y)$, of the payload capability Y of the aircraft, with the assumption that the mission originates at a randomly chosen point in the region at a random time of day.

3. Given a specified probability distribution, $H_X(x)$, of the loads to be carried, determine the probability of mission accomplishment, which is the probability that the payload capability exceeds the load to be carried. Both analytical and graphical methods of solution are given in this paper.

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*This paper is based on work performed by the authors at Boeing Company, Vertol Division.

METHODS FOR DETERMINING ALTITUDE-TEMPERATURE PROBABILITY DISTRIBUTION FOR A GIVEN REGION

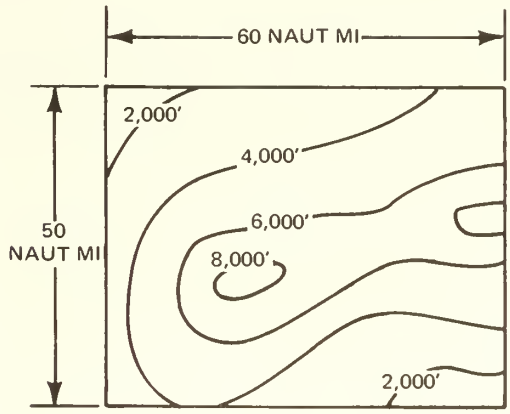
Altitude Probabilities

The first step in determining the altitude-temperature probability mapping is to determine the altitude (elevation) distribution of the area under study by one of the following methods:

1. Use of digital terrain data. The U.S. Army Topological Command has altitude data for certain countries available on magnetic tapes in digital form. The altitude distribution for any region within the country can be derived from the tapes by using sampling techniques. By taking a large sample (in some cases a 100 percent sample) a reasonably continuous distribution for altitude can be obtained. Computer programs for deriving such probability distributions for altitude are described in Reference [4].

2. Use of contour maps. The altitude distribution can also be generated by using a planimeter to measure areas at or below a given altitude on a contour map of the geographical region. (Reference [7]).

Consider the sample geographical region shown in Figure 1. Areas within this region may be measured and converted to altitude probabilities. Figure 2 shows a plot of the cumulative altitude probabilities derived from Figure 1 as a continuous function of altitude; this distribution is defined as $F_A(a)$.



Alt	Area $(Mi)^2$	%	Cum %
0–2,000	360	12	12
2,000–4,000	1,110	37	59
4,000–6,000	1,050	35	84
6,000–8,000	360	12	96
8,000–10,000	120	4	100
TOTAL	3,000	100	100

FIGURE 1

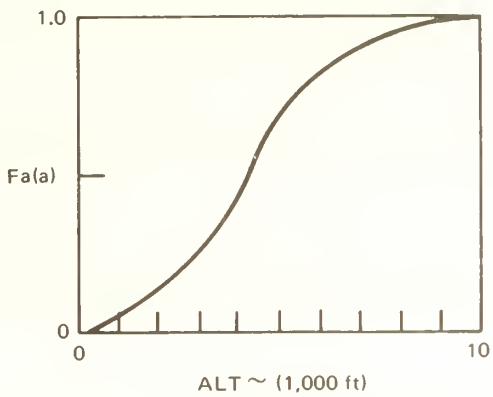


FIGURE 2

The use of contour maps to measure the altitude distribution is less accurate than the use of the digital tapes. However, it gives acceptable results in situations where such digital data or the appropriate programs are not readily available.

If $F_A(a)$ denotes the cumulative distribution function for altitude (i.e., a point chosen randomly from the region has a probability $F_A(a)$ of lying less than altitude a), then we can let $f_A(a)$ represent the probability density function of altitude by,

$$f_A(a) = \begin{cases} \dfrac{dF_A(a)}{da}, & a \geqslant , 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Temperature Distribution

The temperature, T, at any specific altitude is a random variable. The conditional distribution of T at a specified altitude a, $f_{T/A}(t/a)$, can be derived by one of the following methods: (i) Compilation of complete or sampled weather data for the region into an empirical curve; (ii) Application of empirical nomographs described in Reference [5] when only limited weather data are available; (iii) Use of the beta distribution, which uses the same statistics required in (ii) and has been shown to fit very closely the empirical curves used in that method (Reference [3]). Figure 3 summarizes results of some test cases using the beta distribution.

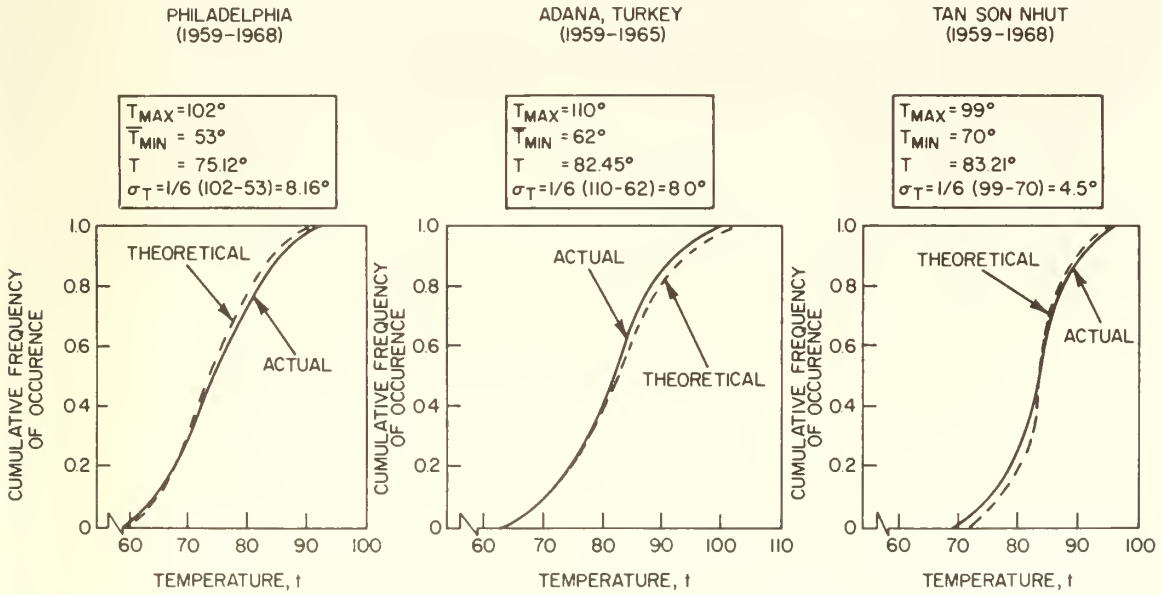


FIGURE 3. Actual *vs* theoretical distribution of temperatures for selected cities that use the beta distribution approach.

Use of Beta Distribution

The distribution is based on these statistics:

T_{max} = Global Maximum = highest temperature ever recorded at the given altitude;
T_{min} = Global Minimum = lowest temperature ever recorded at the given altitude;

$\bar{T}$ = Grand Mean = mean of all temperatures ever recorded at the given altitude;
σ_T = Standard Deviation = $\frac{1}{6}(T_{\max} - T_{\min})$.

If the random variable Z, defined over the interval (0, 1), is a normalized random variable for temperature, then

$$Z = \frac{(T - T_{\min})}{T_{\max} - T_{\min}},$$

follows a beta distribution with mean

$$\mu_Z = \frac{\bar{T} - T_{\min}}{T_{\max} - T_{\min}}$$

and standard deviation

$$\sigma_Z = \frac{\sigma_T}{T_{\max} - T_{\min}} = \frac{1}{6}.$$

The density of Z can then be expressed as

$$f_Z(z; \alpha, \beta) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} & z^{\alpha-1}(1-z)^{\beta-1}, \quad 0 \leq z \leq 1 \\ 0 & \text{elsewhere,} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$ are the parameters of the distribution.

The mean and variance of the distribution are given by

$$E(Z) = \mu_Z = \frac{\alpha}{\alpha+\beta}$$

(2)

$$\text{Var}\,(Z) = \sigma_Z^2 = \frac{\alpha\beta}{(\alpha+\beta)(\alpha+\beta+1)}.$$

By solving Equation (2) for α and β, we get

$$\alpha = \frac{\left[\left(\frac{1-\mu_Z}{\mu_Z}\right) - \left(\frac{1-\mu_Z}{\mu_Z}\right)^2 \sigma_Z^2 - 2\left(\frac{1-\mu_Z}{\mu_Z}\right)\sigma_Z^2 - \left(\frac{1-\mu_Z}{\mu_Z}\right)^2\right]}{\sigma_Z^2\left[\left(\frac{1-\mu_Z}{\mu_Z}\right) + 1\right]^2}$$

(3)

$$\beta = \left(\frac{1-\mu_Z}{\mu_Z}\right)\alpha.$$

The values of α and β at any specified altitude can then be calculated by making the appropriate substitution of

$$\mu_Z = \frac{\bar{T} - T_{\min}}{T_{\max} - T_{\min}} \quad \text{and} \quad \sigma_Z^2 = \frac{1}{36} \text{ into Equation (3).}$$

If the values of $T_{\max}$, $T_{\min}$, and $\bar{T}$ are known at sea level ($a=0$), we can determine the parameters α_0, β_0 for the sea level temperature distribution; i.e.,

$$f_{Z/A}(z|0)=\begin{cases}\dfrac{\Gamma(\alpha_0+\beta_0)}{\Gamma(\alpha_0)\Gamma(\beta_0)}\,z^{\alpha_0-1}(1-z)^{\beta_0-1}, & 0\leq z\leq 1\\ 0, & \text{elsewhere.}\end{cases} \tag{4}$$

Conditional Distribution of Temperature on Altitude

Assume that a standard lapse rate r applies to temperature with increasing altitude for the given geographical area such that

$$t=t^\circ-ra,$$

where t° = temperature at sea level,

a = altitude,

t = temperature at altitude a,

r = change in temperature per foot of altitude (lapse rate).

If $T^\circ_{\max}$, $T^\circ_{\min}$, $\bar{T}^\circ$ are the maximum, minimum, and grand mean temperatures at sea level, and $\sigma^\circ_T=1/6(T^\circ_{\max}-T^\circ_{\min})$ is the standard deviation, then the corresponding values at altitude a will be

$$\begin{aligned}T^a_{\max}&=T^\circ_{\max}-ra\\ T^a_{\min}&=T^\circ_{\min}-ra\\ \bar{T}^a&=\bar{T}^\circ-ra\\ \sigma^a_T&=1/6(T^a_{\max}-T^a_{\min})=1/6(T^\circ_{\max}-T^\circ_{\min})=\sigma^\circ_T.\end{aligned} \tag{5}$$

The normalized mean temperature and standard deviation at altitude a are then

$$\begin{aligned}\mu^a_Z&=\frac{\bar{T}^a-T^a_{\min}}{T^a_{\max}-T^a_{\min}}=\frac{\bar{T}^\circ-T^\circ_{\min}}{T^\circ_{\max}-T^\circ_{\min}}=\mu^0_Z\\ \sigma^a_Z&=\frac{\sigma^a_T}{T^a_{\max}-T^a_{\min}}=\frac{\sigma^0_T}{T^\circ_{\max}-T^\circ_{\min}}=\sigma^0_Z=1/6.\end{aligned} \tag{6}$$

Therefore the values α_a and β_a for the distribution of the temperature at altitude a are the same as α_0 and β_0, respectively. In other words, the distribution of the normalized temperature remains the same at any altitude under the assumption of a standard lapse rate.

By normalizing Z to the interval (0, 1) for any altitude a, we have

$$Z=\frac{T-T^a_{\min}}{T^a_{\max}-T^a_{\min}}=\frac{T-T^\circ_{\min}+ar}{T^\circ_{\max}-T^\circ_{\min}},$$

where the distribution of Z for any altitude a is;

$$f_{Z/A}(z/a)=\begin{cases}\dfrac{\Gamma(\alpha_0+\beta_0)}{\Gamma(\alpha_0)\,\Gamma(\beta_0)}z^{\alpha_0-1}(1-z)^{\beta_0-1}, & 0\leq z\leq 1\\ 0, & \text{elsewhere.}\end{cases}$$

Now, given the distribution $f_{Z/A}(z/a)$, we attempt to solve for the distribution of $f_{T/A}(t/a)$.

Solving for T in terms of Z,

$$T=g(Z)=Z(T^\circ_{\max}-T^\circ_{\min})+T^\circ_{\min}-ar.$$

The distribution of the actual temperature T, given an altitude a, is given by

$$(9)\qquad f_{T/A}(t/a)=f_{Z/A}\left[g^{-1}(t)\mid a\right]\left|\frac{dg^{-1}(t)}{dt}\right|=\frac{1}{T^\circ_{\max}-T^\circ_{\min}}f_{Z/A}\left\{g^{-1}(t)\mid a\right\}$$

$$=\begin{cases}\dfrac{1}{T^\circ_{\max}-T^\circ_{\min}}\dfrac{\Gamma(\alpha_0+\beta_0)}{\Gamma(\alpha_0)\,\mathrm{P}(\beta_0)}\left[\dfrac{t-T^\circ_{\min}+ar}{T^\circ_{\max}-T^\circ_{\min}}\right]^{\alpha-1}\left[\dfrac{T^\circ_{\max}-t-ar}{T^\circ_{\max}-T^\circ_{\min}}\right]^{\beta-1} & \text{when } a>0\\ & \text{and } T^\circ_{\min}\leq ar+t\leq T^\circ_{\max}\\ 0, & \text{elsewhere.}\end{cases}$$

The Joint Density Function of Altitude and Temperature

Using Bayes theorem, we have

$$(10)\qquad f_{T/A}(t/a)=\begin{cases}f_{T,A}(t,a)/f_A(a), & a>0,\ T^\circ_{\min}\leq t+ar\leq T^\circ_{\max};\\ 0, & \text{elsewhere}\end{cases}$$

Hence,

$$(11)\qquad f_{T,A}(t,a)=\begin{cases}f_{T,A}(t/a)f_A(a), & a>0,\ T^\circ_{\min}\leq t+ar\leq T^\circ_{\max};\\ 0, & \text{elsewhere.}\end{cases}$$

Graphic Display of Temperature-Altitude Distribution

Figure 4 shows the relation between t, altitude, a, and sea level temperature, $t°$, assuming a standard lapse rate r. The dotted line shows how the equivalent sea-level temperature 110.77° is obtained when the temperature is 100° at 3,000 ft, with a lapse rate of $r=0.00359°/\text{ft}$.

Figure 5 shows how the conditional probability distribution of the temperature, t, at any altitude, a, can be determined from the fact that the cumulative distribution of the normalized temperature remains the same at any altitude. In this case, the dotted lines show that the probabilities of attaining temperatures less than 110.77° at sea level and less than 100° at 3,000 ft are the same at 0.82.

Figure 6 shows a linear display of the cumulative probability of altitude occurrence. Cumulative probabilities are shown on a linear scale on the right and corresponding altitudes are scaled on the left. This data may be obtained directly from Figure 2. The display in Figure 6 in effect shows altitude intervals physically proportional to their actual frequency of occurrence.

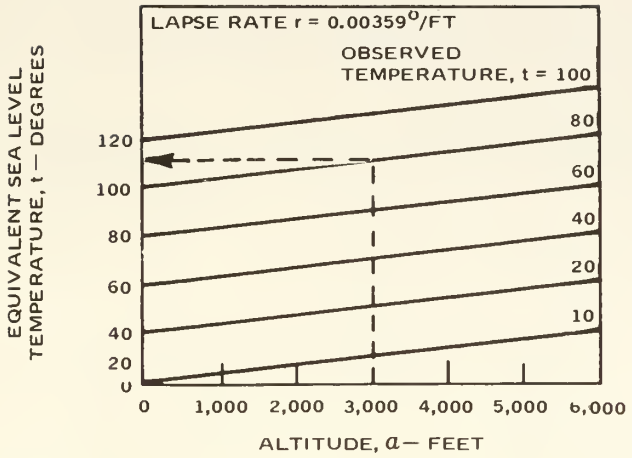
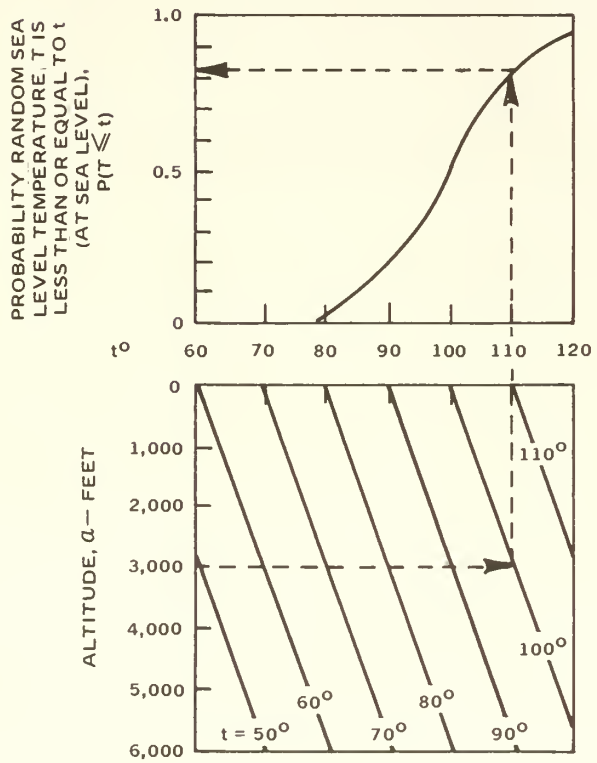


FIGURE 4. Sea level temperature equivalent to temperatures at altitude.



NOTE: THE EXAMPLE SHOWN BY THE DOTTED LINES INDICATES THAT THE PROBABILITY OF A TEMPERATURE LESS THAN OR EQUAL TO 100°F AT 3,000 FEET IS APPROXIMATELY 0.82.

FIGURE 5. Determining conditional probabilities of temperatures at given altitudes.

A linear probability scale for temperature given altitude is then added along the lower horizontal axis. Next, temperatures associated with cumulative probabilities at different altitudes may be cross-plotted from Figure 5 to Figure 7. One such point, 100° at 3,000 ft with a probability of 0.82 is shown on Figure 7. Once all temperature points are appropriately spotted on the respective altitude lines, all points are joined into isothermal lines. The result is a complete temperature-altitude probability mapping as shown in Figure 7. Areas within the graph in Figure 7 represent actual probabilities; e.g., the probability that the altitude is between 3,000 and 4,000 ft and the temperature between 90 and 100° is equal to the proportion of the shaded area to the entire area.

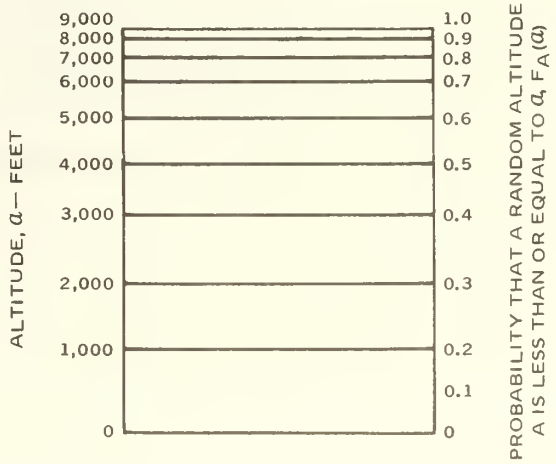


FIGURE 6. Linear display of cumulative probability of altitude occurrence.

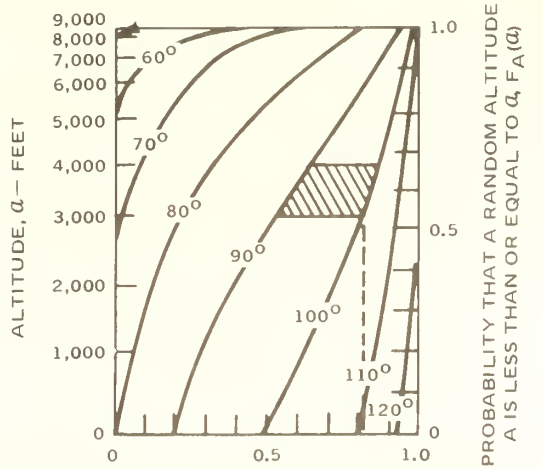


FIGURE 7. Complete altitude-temperature probability mapping.

METHODS FOR DETERMINING PROBABILITY DISTRIBUTION FUNCTION OF PAYLOAD CAPABILITY

The mission for the helicopter is to carry a load over a fixed mission radius. The available payload (or payload capability) can be calculated from aerodynamic considerations by the equation:

$$y = W_{TO}(a, t) - [M_f(a, t, m_r) + W_e + W_f], \tag{12}$$

where

y = maximum available payload that the helicopter can carry for the mission;
a = mission altitude;
t = mission temperature;
m_r = mission radius;
$W_{TO}(a, t)$ = maximum takeoff gross weight of helicopter; i.e., that weight where power required in hover matches power available at a and t;
$M_f(a, t, m_r)$ = mission fuel required at a, t, and m_r;
W_e = empty weight of helicopter; and
W_f = weight of fixed equipment.

Briefly, values of $W_{TO}(a, t)$ are determined by (i) Finding what power is available at (a, t) to the helicopter's gas turbine engine(s) (this information is available from the engine manufacturer), and (ii) Using helicopter aerodynamic equations (Reference [2]), to solve for the maximum helicopter gross weight, W_{TO}, that will hover with the power determined by (i). Mission fuel $M_f(a, t, m_r)$ may also be computed from standard performance equations. Since all variables in (12) are fixed at a given mission radius except for a and t, we will hereafter refer to the expression for available payload as,

$$y = \phi(a, t). \tag{13}$$

A typical surface depicting $\phi(a, t)$ is shown in Figure 8.

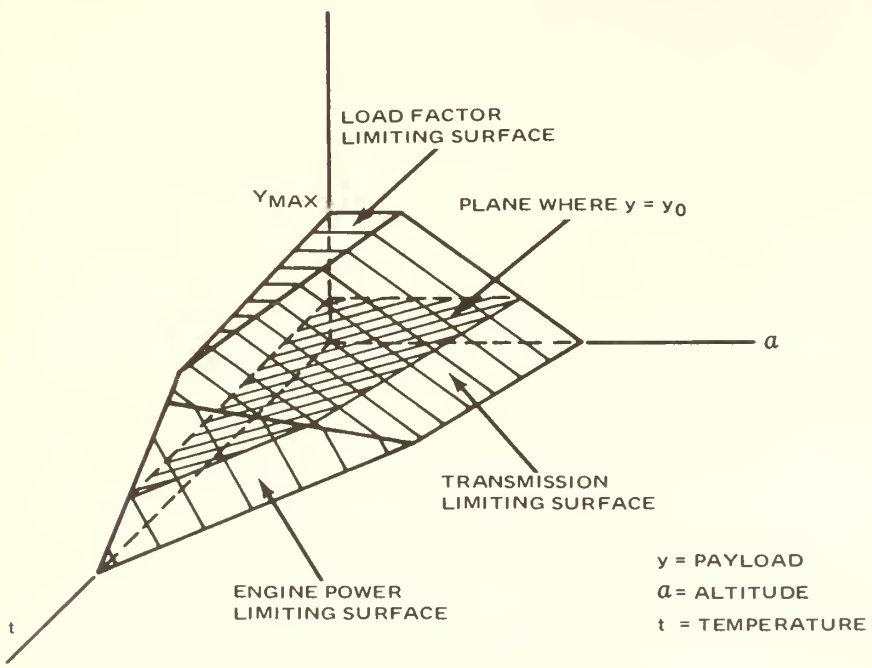


FIGURE 8. Payload as a function of altitude and temperature.

Analytic Approach

Assume that a given region has associated with it an altitude-temperature joint probability density function, $f_{A,T}(a,t)$.

The available payload y at any altitude-temperature condition is given by (13), where y is computed from standard helicopter performance equations, varying a and t only. As seen in Figure 8, y is bounded above by y_{max} (which is limited by the load factor) and below by $y_{min}=0$.

The problem now is to find the distribution $F_Y(y)$ of available payload Y, where $Y=\phi(A,T)$, i.e., a function of the random variables A and T. Consider integrating under $f_{A,T}(a,t)$ in the a,t plane for values of y, where $y \geq y_0$ in Equation (13) (see Figure 9).

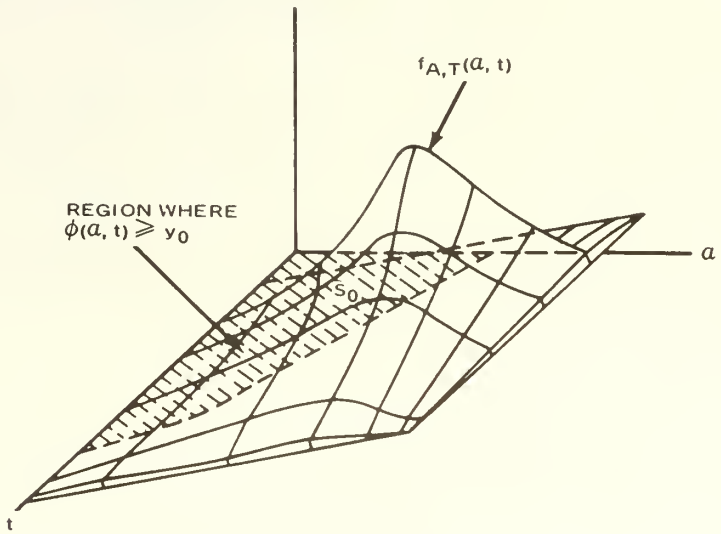


FIGURE 9. Region of integration under $f_{A,T}(a,t)$ to determine $1-F_Y(y_0)$ (i.e., the probability that a random load $Y=r(A,T)\geq y_0$).

The resulting integration will represent the probability that available payload is at least y_0. Let S_0 be the region in which $\phi(a,t)\geq y_0$. Then

$$P(Y \geq y_0) = \int_{S_0}\int f_{A,T}(a,t)\,da\,dt. \tag{14}$$

Graphic Solution to Probability of Carrying a Given Payload

The function of $y=\phi(a,t)$ show in Figure 8 can be shown in parametric form as in Figure 10. Consider the vertical dashed line on Figure 10 representing the payload y_0. Each parametric temperature line that this line crosses defines an altitude where a payload of at least y_0 is available. The horizontal line shows that at 90° and 4,200 ft, y_0 may be attained (Point Q). The point Q (90°, 4,200 ft) is shown cross-plotted onto the temperature-altitude probability mapping given in Figure 11.

By cross-plotting all altitude-temperature pairs for the constant payload y_0 onto Figure 11, $F_Y(y_0)$ can be determined by measuring the areas to the right of the constant $y=y_0$ line (i.e., the shaded area in Figure 11). The resulting value of $F_Y(y_0)$ is then cross-plotted onto a graph of y vs $F_Y(y_0)$.

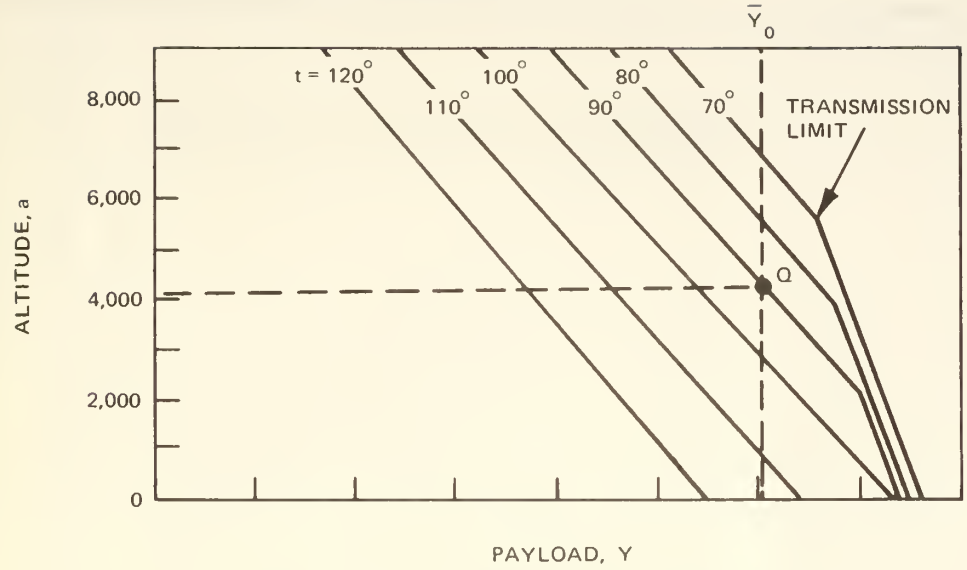


FIGURE 10. Parametric display of payload capability.

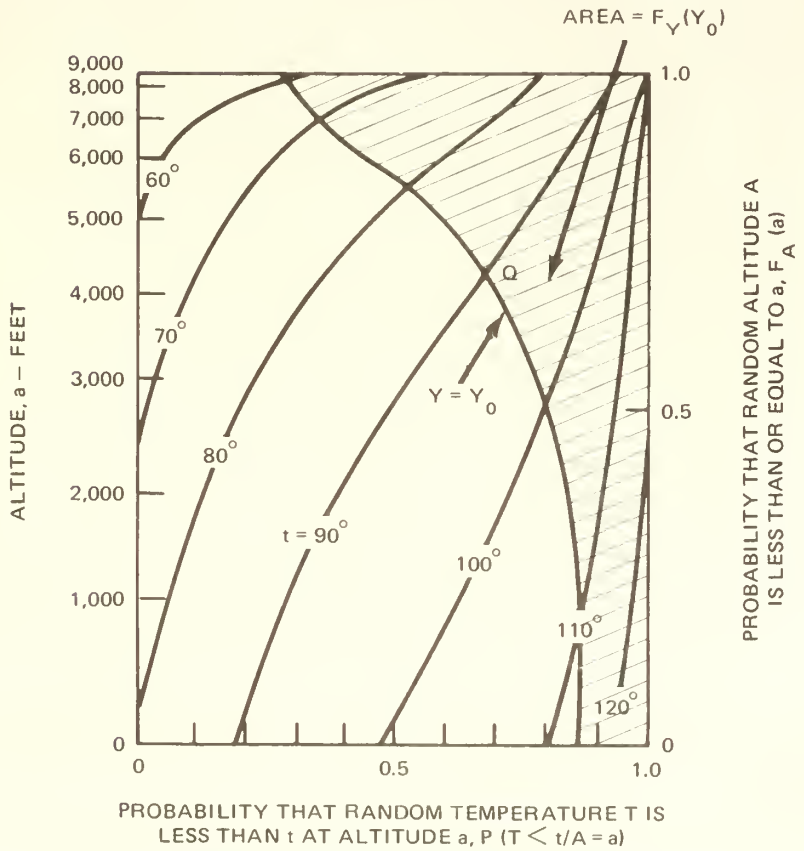


FIGURE 11. Payload on altitude-temperature mapping.

By determining values of $F_Y(y_0)$ for all values of y, a cumulative distribution can be constructed as in Figure 12.

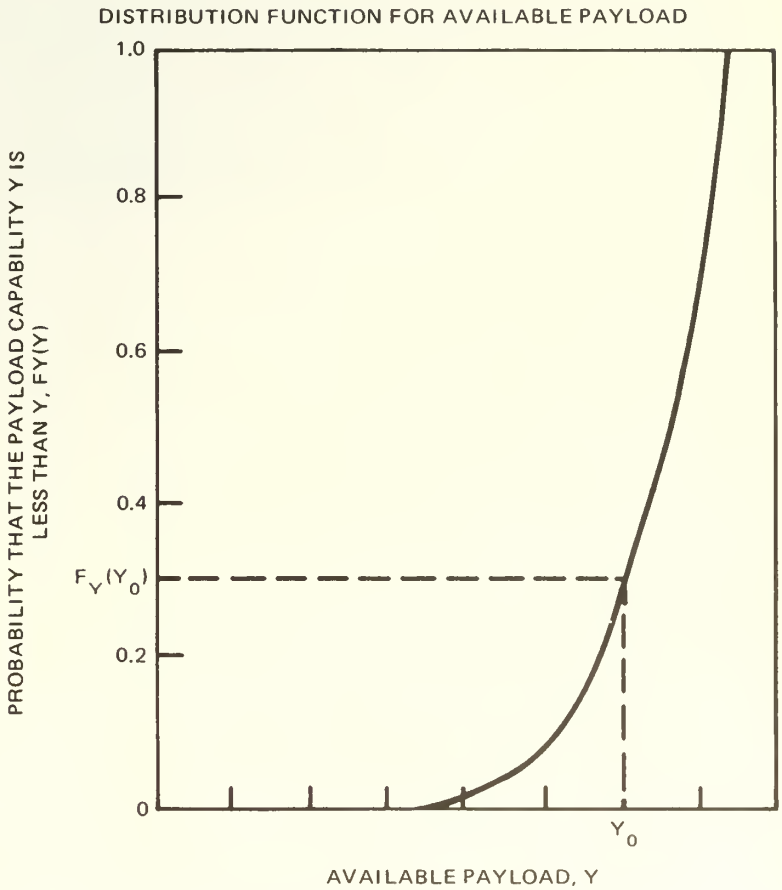


FIGURE 12. Distribution function for available payload.

Define

$$F_Y(y_0) = P(Y < y_0).$$

Then

$$P(Y \geq y_0) = 1 - F_Y(y_0) = \int_{S_0} \int f_{A,T}(a, t)dadt,$$

and

$$F_Y(y_0) = 1 - \int_{S_0} \int f_{A,T}(a, t)dadt. \tag{15}$$

DETERMINATION OF PROBABILITY OF MISSION ACCOMPLISHMENT (PMA)

Analytic Method

Let X be a random variable that describes the required payload and let $H_X(x)$ be its distribution function (see Figure 13).

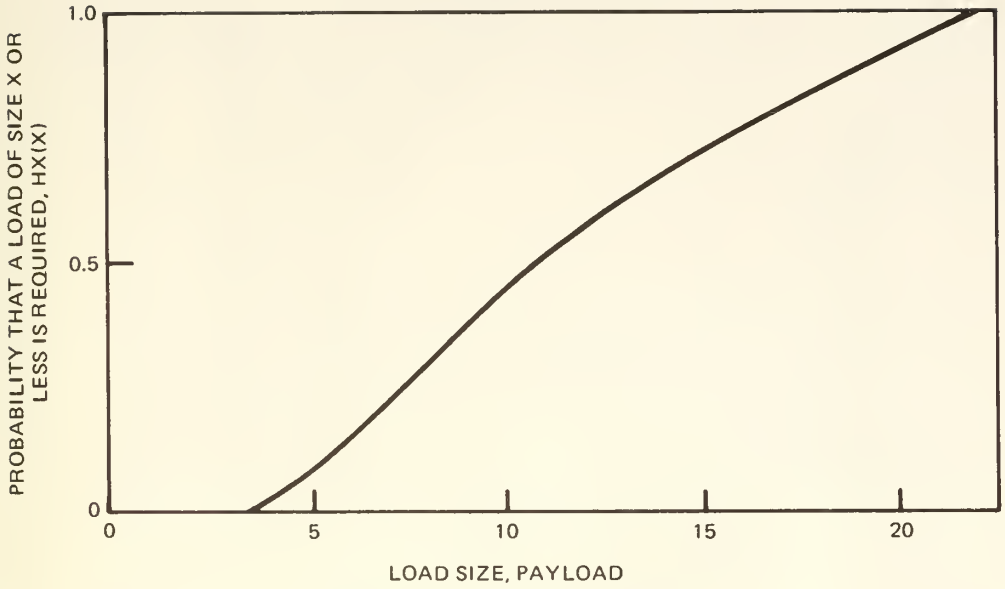


FIGURE 13. Load distribution function.

Also let MA signify the event Mission Accomplishment or the event $Y \geq X$ (available payload greater than required payload). From the previous section the distribution function of available payload is $F_Y(y)$. The conditional probability of Mission Accomplishment given a required payload is

$$P(MA|X=x)=P(Y\geq x|X=x)=\int_{y=x}^{\infty} dF_Y(y)=1-F_Y(x).$$

The probability of mission accomplishment (PMA) over all possible values of x is then

$$P(MA)=\int_{x=0}^{\infty} P(MA|X=x)dH_X(x)=\int_{x=0}^{\infty}[1-F_Y(x)]dH_X(x). \tag{16}$$

Graphic Method

A graphic method for determining PMA can be achieved by the following steps (see Figure 14):

1. Plot values of load requirements x directly across from corresponding values of $H_X(x)$, where $H_X(x)$ is plotted on a linear scale.

2. Cross-plot the value of $1-F_Y(x)$ in the y-axis direction for each x with values of $1-F_Y(x)$ on a linear scale. PMA is represented by the unshaded area in Figure 14.

APPLICATION OF THE METHODOLOGY

This methodology was used recently to evaluate the capability of helicopters performing a container transport mission in different parts of the world. The distribution of container weights was obtained from the shipping reports (Reference [8]) for the 8 by 8 by 20-ft containers shipped from the west coast of the United States to the Far East. The probability of mission accomplishment was computed for a number of aircraft with different design payloads to determine the tradeoff between the probability of not accomplishing the mission and the cost of the aircraft.

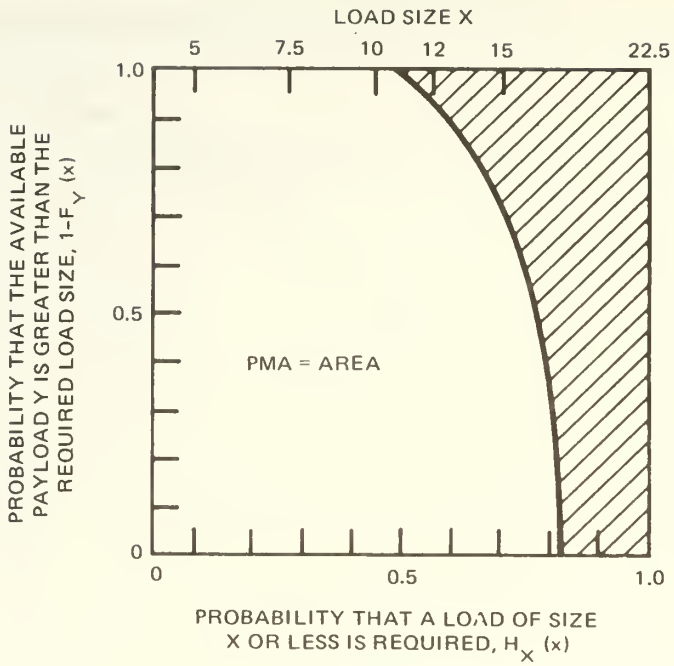


FIGURE 14. Graphic determination of probability of mission accomplishment (PMA).

Aircraft were evaluated in the environment of southeast Asia using the data for the hottest month. Four hover conditions were examined: hover in ground effect at 15 ft (HIGE 15 ft); hover in ground effect at 50 ft (HIGE 50 ft); hover out of ground effect (HOGE); and hover with sufficient power reserve to provide a 500-fpm vertical rate of climb (500 FPM VRC).

Design payloads ranging from 20T to 24T were examined under the same conditions. The information was summarized by comparing the probability of not accomplishing the mission with the system cost of the aircraft, as shown in Figure 15. It was concluded that a payload of 23 tons appeared to be the largest design worth considering, since the marginal decrease in probability of not accomplishing the mission was not sufficient to justify the increased cost.

CONCLUSION

The methodology provides a technique for rapidly evaluating the trade-off between increased costs of larger helicopters against the reduced probability of not accomplishing the mission. Additionally, the graphic approach provides an effective display of the results. The overall approach could also prove very useful in other situations where the parameter under evaluation is a function of two jointly distributed variables.

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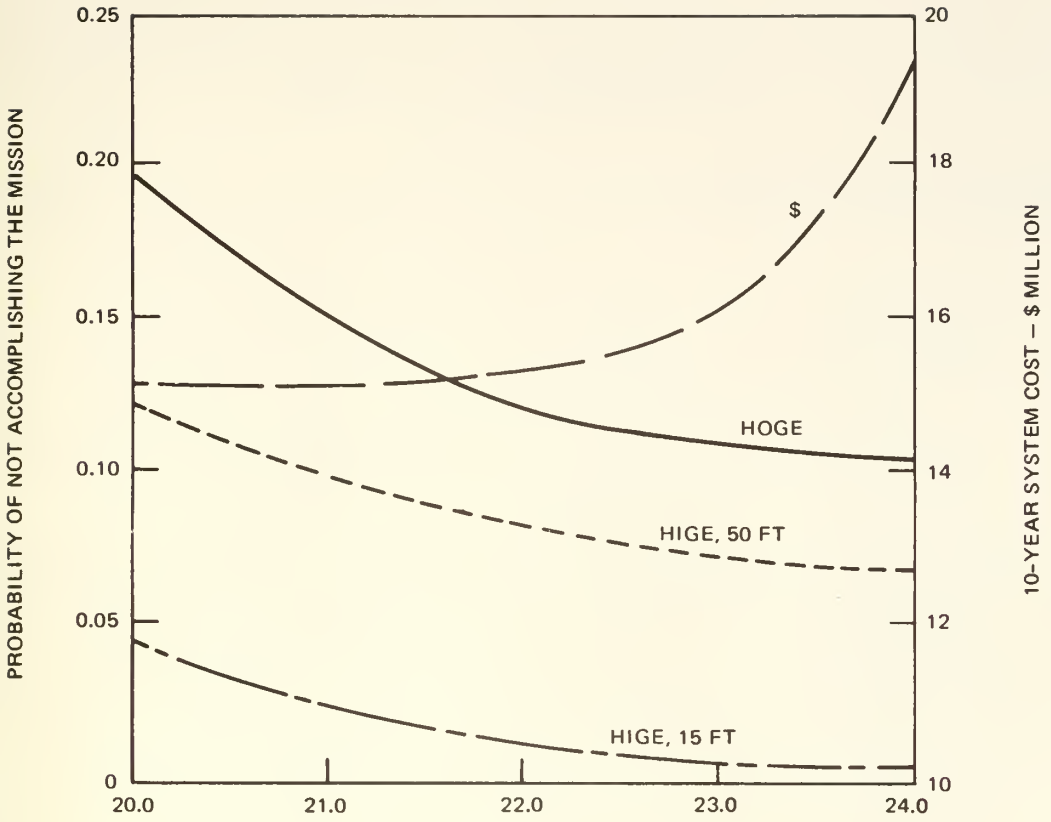

FIGURE 15. Application of methodology—A cost effectiveness study.

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(NOTE: The Boeing Documents are available from the library of the Vertol Division, Boeing Company, Philadelphia)

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